TSUNAMI RUNUP IN U AND V SHAPED BAYS

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A

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Abstract

Tsunami runup can be effectively modeled using the shallow water wave equations. In 1958 Carrier and Greenspan in their work "Water waves of finite amplitude on a sloping beach" used this system to model tsunami runup on a uniformly sloping plane beach. They linearized this problem using a hodograph type transformation and obtained the Klein-Gordon equation which could be explicitly solved by using the Fourier-Bessel transform. In 2011, Efim Pelinovsky and Ira Didenkulova in their work "Runup of Tsunami Waves in U-Shaped Bays" used a similar hodograph type transformation and linearized the tsunami problem for a sloping bay with parabolic cross-section. They solved the linear system by using the D'Alembert formula. This method was generalized to sloping bays with cross-sections parameterized by power functions. However, an explicit solution was obtained only for the case of a bay with a quadratic cross-section.

In this paper we will show that the Klein-Gordon equation can be solved by a spectral method for any inclined bathymetry with power function for any positive power. The result can be used to estimate tsunami runup in such bays with minimal numerical computations. This fact is very important because in many cases our numerical model can be substituted for fullscale numerical models which are computationally expensive, and time consuming, and not feasible to investigate tsunami behavior in the Alaskan coastal zone, due to the low population density in this area.

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Introduction

Today tsunami waves are the object of intense study. These hazardous waves, caused mostly by earthquakes and volcano eruptions, transfer tremendous amount of energy. When they approach the shore, these waves become dangerous due to the amount of energy that they carry. This energy gets released on the land causing to serious destructions made by water. As they approach the shoreline, the waves' height becomes very large and as a result, the coastal land becomes flooded. Sometimes the water can travel far inland bringing large damages to property and ecosystems, as well as loss of human lives. (That is why scientists try to understand this phenomenon so that it can be described and in the future these losses can be minimized.)

One of the directions of study is obtaining and analyzing physical models of tsunami waves near the shore. When a tsunami wave approaches the shore its length becomes very large in comparison to the depth of the water. As a result, nonlinear models like the shallow water wave equations can be used. In most of cases there is no explicit algorithm for solving the nonlinear equations. As a consequence, one of the most effective approaches to solve the nonlinear equations is to transform the nonlinear equations to linear ones. The process of making the nonlinear equations linear is often called a linearization. We will linearize the shallow water wave equations. In order to be able to do this, we will make special assumptions on the wave shape, the direction of propagation, and the bathymetry of the bay. In previous studies, these assumptions were the following: water approached the shore uniformly, the viscosity of water was neglected, the bay had a sloping bathymetry and the cross-section had a specific form described by an even function. In (*Carrier and Greenspan*, 1958) using a hodograph type transformation, the authors analytically solved the wave runup problem for a wave with a uniformly approaching water front in a plane beach with constant slope. In (*Didenkulova and Pelinovsky*, 2011) using the same type of transformation, the authors analytically solved a similar problem but for an inclined bay with sloping bathymetry and parabolic cross-section. There have been no other publications about wave runup problems with other bathymetries. In our problem we use a similar hodograph type transformation to obtain analytical solutions for the runup problem in bays with an inclined bathymetry and a cross-section parameterized by a power function for any positive degree. Using this solution we show that it produces the same results as those, derived for a plane sloping beach and inclined sloping bay with parabolic cross-section. Then using our solution, we investigate a sloping bay with cross-section parameterized by a power function of the first degree (triangular V shaped bay). Next, referring to *Courant and Hilbert* (1989), we extend the class of bays which have a D'Alembert solution for the wave runup problem and then, using this approach, we investigate the wave runup problem for a V shaped sloping bay with a cross-section described by power function of 2/3 degree, we call these bays 2/3-bays. We produce an explicit solution for the linearized problem for these bays and explicit formulas for the physical wave characteristics for a 2/3-bay, but the practical computation of physical characteristics for both bays will require numerical integration and interpolation techniques.

Chapter 1
Description of the problem

In this paper we will study tsunami wave runup. It includes the behavior of water near the shore. In practice tsunami waves are very long and because we are going to study them near shore, the ratio of the water depth to the length of the wave is very small. This is a reason to use the so called shallow water wave model - a nonlinear model describing the water behavior. The ratio of the water depth to the length of a wave for simplicity is neglected in this model. For a wave running in a one-dimensional space, so that vertical and horizontal water displacements change only along one direction, this model is described by the following system of equations (*Johnson*, 1997):

$$\begin{cases} u_t + uu_x + g\eta_x = 0, \\ S_t + (Su)_x = 0, \end{cases} \tag{1.1}$$

where

- Ox - is the axis set up along the direction of the running wave
- $\eta(x,t)$ - $[m]$ is the height of the vertical water displacement above the usual level at a point (x,y) $\big($it clearly does not depend on $y\big)$
- $u(x,t)$ - $[\frac{m}{s}]$ - is the velocity of the horizontal water displacement
- $g \approx 9.81$ - $[\frac{m}{s^2}]$ - is the gravitational acceleration
- $S(x,t)$ - $[m^2]$ is the area of cross-section of a water body, in the direction perpendicular to a water flow

Let us fix the mutually orthogonal directions Ox, Oy and Oz, so that Oz is the vertical direction. Let us for simplicity consider a bay with a constant slope along the Ox direction so that Ox is directed onshore, as shown in Figure 1.1. Let the bay have a cross-section that can be parameterized by $z = c|y|^m$, where $c > 0$, and $m \in (0,\infty]$ up to vertical translation along the Oz . Let the origin correspond to the initial point of intersection of the shoreline (on the edge of the undisturbed water surface) and a point of symmetry of the bay, so that the bay has a symmetry with respect to Oxy plane (see Figure 1.3). Let $\alpha > 0$ be the value of that slope of the bottom along the Ox direction. Let $z(x,y)$ be the function that represent the bottom surface. Then $z(x,y)$ can be expressed by the formula

$$z(x,y) = \alpha x + c|y|^m, \text{ (see Figure 1.3).}$$

Let us investigate how the cross-section depends on the choice of m. Notice that for the case $m = \infty$, the bay becomes a 2 meter-wide sloping plane beach (Also, notice that to parameterize a beach of arbitrary width, we need just put a constant c inside the absolute value brackets), for $m \in (1,\infty)$ the bottom is smooth (such bays together with the plane beach are called U shaped

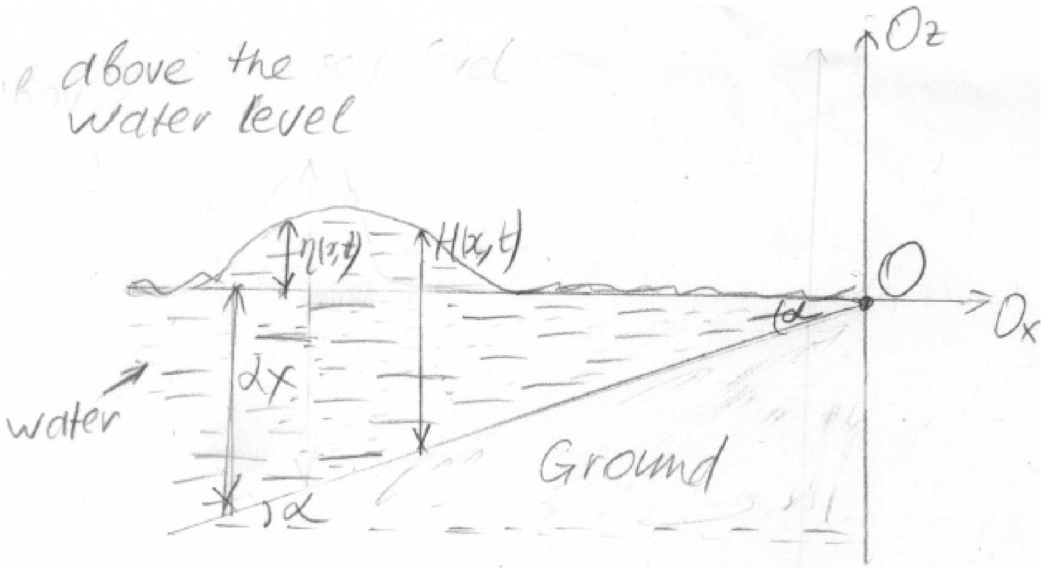


Figure 1.1: View of the wave and Ox axis direction

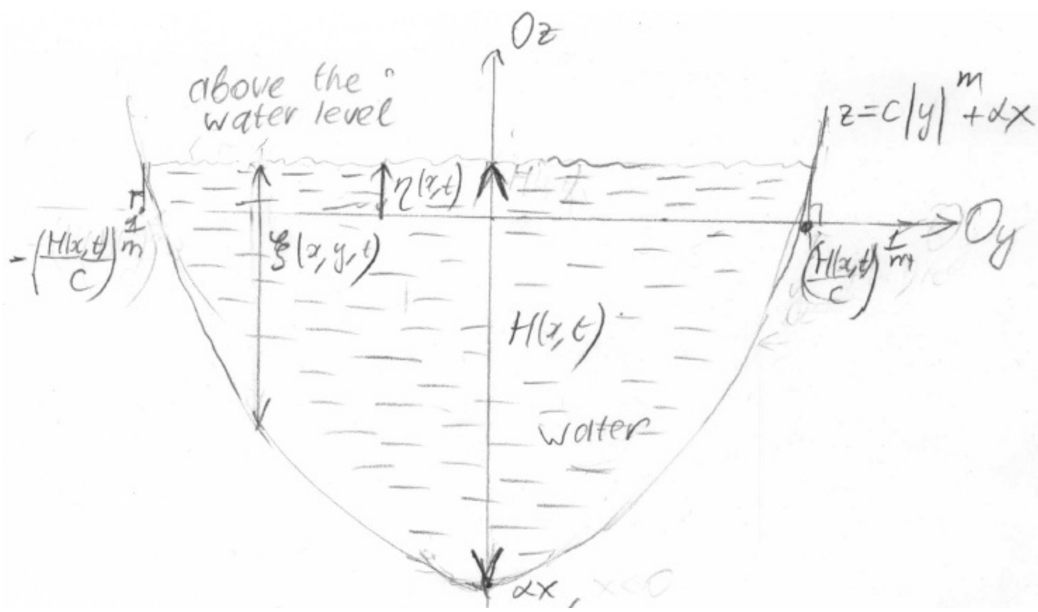


Figure 1.2: Cross-section of a water body and its dependence on the water level and a type of a bay

bays), for $m = 1$ the cross-section becomes triangular, for $m \in (0, 1)$ the bottom has a narrowing trough along Ox direction (these bays together with the triangular cross-section bay are called V shaped bays). The examples of the cross-sections for each type of bay are given in Figure 1.4.

Let a long tsunami wave be running along the Ox direction onto the shore so that the vertical and horizontal water displacements are constant along the cross-section. Then, as one can see from

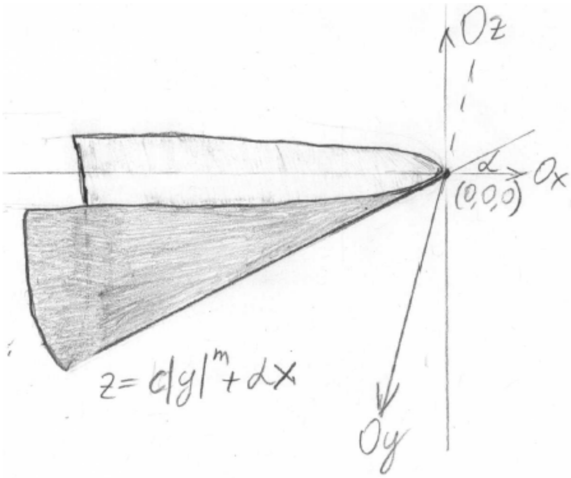
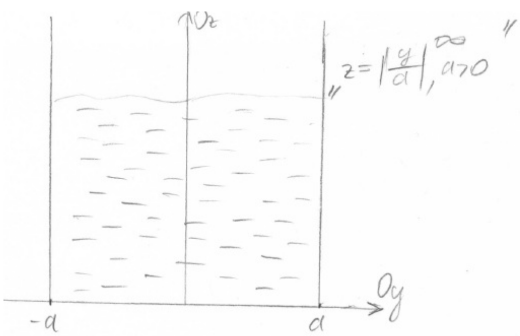
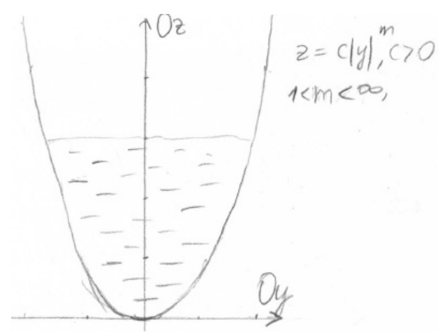


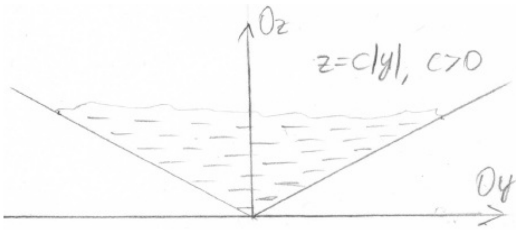
Figure 1.3: Surface of a bay



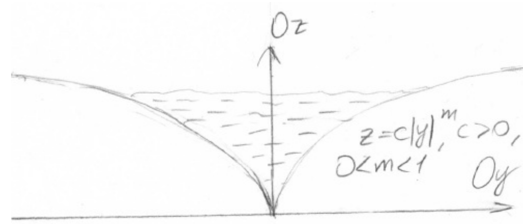
a) plane beach



b) U shaped bay



c) Triangle bay



d) V shaped bay

Figure 1.4: Bays with different types of cross-sections

Figure 1.2, cross-section area S does not depend on y and therefore, the system of shallow water equations (1.1) can be applied to model this wave. Let us for simplicity consider such wave and model it with the system (1.1).

Now we will simplify system (1.1) for our chosen class of bays. In order to simplify the system (1.1) we introduce some new variables. Let $\xi(x, y, t)$ be the water depth at a point (x, y) (see Figure

1.2). It follows that

$$\xi(x,y,t) = \max\{\eta(x,t) - z(x,y), 0\}$$

and then,

$$\xi(x,y,t) = \max\{\eta(x,t) - \alpha x - c|y|^m, 0\}.$$

Let $H(x,t)$ be the maximal water depth for a cross-section water body at a fixed x and t. Then $H(x,t)$ will be a depth measured along the symmetry axis of bay and it can be described by the formula

$$H(x,t) = \eta(x,t) - \alpha x. \tag{1.2}$$

Applying this result to the water depth ξ, we have that

$$\xi(x,y,t) = \max\{H(x,t) - c|y|^m, 0\}. \tag{1.3}$$

Now, in order to simplify (1.1) we will express $S(x,t)$ in terms of $H(x,t)$. First, we note that,

$$S(x,t) = \int_{-\infty}^{\infty} \xi(x,y,t)dy = \int_{-\infty}^{\infty} \max\{H(x,t) - c|y|^m, 0\}dy,$$

and ξ has a finite support. Now, let us find the boundary points for the support of $\xi(x,y,t)$. By the formula (1.3), the boundary of the support is the set of points solving equation

$$H(x,t) = c|y|^m.$$

Solving this equation with respect to y, we have that

$$y = \pm(H(x,t)/c)^{1/m}.$$

As one can see from Figure 1.2, for every fixed x and t the function, $\xi(x,y,t)$ is equal to $H(x,t) - c|y|^m$ for $y \in [-(H(x,t)/c)^{1/m}, (H(x,t)/c)^{1/m}]$ and vanishes outside this interval. Then

$$\begin{aligned} S(x,t) &= \int_{-(H(x,t)/c)^{1/m}}^{(H(x,t)/c)^{1/m}} H(x,t) - c|y|^m dy \\ &= \frac{2m}{m+1} c^{-1/m} H(x,t)^{\frac{m+1}{m}}. \end{aligned}$$

Plugging this result into (1.1) gives us the system

$$\begin{cases} H_t + uH_x + \frac{m}{m+1} Hu_x = 0, \\ u_t + uu_x + g\eta_x = 0. \end{cases}$$

Applying result (1.2) we can rewrite this system as

$$\begin{cases} H_t + uH_x + \frac{m}{m+1} Hu_x = 0, \\ u_t + uu_x + gH_x = -g\alpha. \end{cases} \tag{1.4}$$

So, the tsunami wave with our specific choices of a bay and wave front is modeled by system of two equations. The number of variables in this system of equations was reduced down to 2, and that makes this system looking simpler.

In order to apply this model to a wave, we need to know some information about the wave. At least we need to know the initial shape of the water surface (which can be expressed through H) and its velocity u for some fixed instant of time. Let this instant of time be 0. The next assumptions which we will use for the wave is that at the initial time instant the velocity of the horizontal water displacement u is 0 and we know the shape of the wave at that instant. Let it be expressed through H at initial time instant as $H(x, 0)$. Let it be some function $\phi(x)$ defined for $x < 0$ (since the Ox axis is directed onshore). So, for a wave we have that

$$\begin{cases} u(x,0) = 0, \\ H(x,0) = \phi(x), \end{cases} \tag{1.5}$$

where $\phi(x)$ is known and given by the wave shape. Equations (1.5) form initial conditions for the system (1.4). In practice it is easy to express the initial shape of the wave as $\eta(x, 0)$. In this case the formula (1.2) lets us convert $\eta(x, 0)$ into $H(x, 0)$.

Next, we would like to consider the boundary conditions for our system. Since the boundary of the flooded area moves with the fluid flow, it does not seem possible to give it in physical coordinates for the edge of the flooding area. But the reasonable condition of this boundary would be the finiteness of the velocity u and $H(x, t)$ be equal to zero. Considering the offshore direction it is important to notice that every physical wave decays and vanishes. Then the reasonable boundary conditions would be

$$\begin{cases} \lim_{x \to -\infty} u(x,t) = 0, \\ \lim_{x \to -\infty} \eta(x,t) = 0, \end{cases} \tag{1.6}$$

where $H(x, t)$ depends on $\eta(x, t)$ and this dependence is showed by the formula (1.2).

Chapter 2

Linearization of the system of shallow water equations

In order to find physical characteristics of a modeled wave we need to solve the shallow water equation (1.4) with the stated initial conditions (1.5) and natural boundary conditions (1.6). As one can see, the system (1.4) is nonlinear and to able to solve it somehow, we will first linearize it, reproducing the procedure published in (*Zahibo et al.*, 2006). to find an optimal and precise way to solve it, we would like to linearize it.

The method we use for it is the method of characteristic curves (*Farlow*, 1993). The brief idea of this method is to make the independent variables x and t dependent and therefore, to make the variables u and H be independent.

2.1 Method of characteristics

First, with a purpose making the work with the system 1.4 easier, we will write this system in a matrix form. In the matrix form this system is

$$\begin{pmatrix} H \\ u \end{pmatrix}_t + \begin{pmatrix} u & \frac{m}{m+1}H \\ g & u \end{pmatrix} \begin{pmatrix} H \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ -g\alpha \end{pmatrix} \tag{2.1}$$

Let

$$M = \begin{pmatrix} u & \frac{m}{m+1}H \\ g & u \end{pmatrix}.$$

Let us solve this system by characteristic curves. Let us introduce a parameter s and consider a characteristic curve $(x(s), t(s))$ (let us call this curve as C). Then the rate of change of H and u along this curve can be expressed through the following matrix formula:

$$\frac{d}{ds}\begin{pmatrix} H \\ u \end{pmatrix} = \frac{dx}{ds}\begin{pmatrix} H \\ u \end{pmatrix}_x + \frac{dt}{ds}\begin{pmatrix} H \\ u \end{pmatrix}_t$$

Using the system (2.1) and the introduced matrix M, and making some simplification we have that rate of change of H and u along the curve C can be expressed as

$$\frac{d}{ds}\begin{pmatrix} H \\ u \end{pmatrix} = \left(I\frac{dx}{ds} - M\frac{dt}{ds}\right)\begin{pmatrix} H \\ u \end{pmatrix}_x + \begin{pmatrix} 0 \\ -\alpha g \end{pmatrix}\frac{dt}{ds}.$$

There is an important fact: the matrix $\left(I\frac{dx}{ds} - M\frac{dt}{ds}\right)$ must be singular. Otherwise, we could find the partial derivatives of H and u with respect to x from the knowledge of H and u on C. Recall that the matrix is singular if and only if its determinant is 0. Therefore, $det\left(I\frac{dx}{ds} - M\frac{dt}{ds}\right) = 0$.

Now let us find a formula for this determinant. Clearly,

$$det\left(\left(I\frac{dx}{ds} - M\frac{dt}{ds}\right)\right) = \left(\frac{dx}{ds} - u\frac{dt}{ds}\right)^2 - \frac{m}{m+1}gH\left(\frac{dt}{ds}\right)^2.$$

Next, since the determinant equal 0,

$$\left(\frac{dx}{ds} - u\frac{dt}{ds}\right)^2 - \frac{m}{m+1}gH\left(\frac{dt}{ds}\right)^2 = 0,$$

and solving this equation with respect to $\frac{dx}{ds}$ we obtain that

$$\frac{dx}{ds} = \left(u \pm \sqrt{\frac{m}{m+1}gH}\right)\frac{dt}{ds}.$$

So, we have 2 characteristic curves. Let us denote them C_+ and C_- so that $\frac{dx}{ds} = \left(u + \sqrt{\frac{m}{m+1}gH}\right)\frac{dt}{ds}$ along C_+ and $\frac{dx}{ds} = \left(u - \sqrt{\frac{m}{m+1}gH}\right)\frac{dt}{ds}$ along the curve C_- respectfully. Let us for simplicity introduce the variables λ_+ and λ_- so that

$$\lambda_\pm = u \pm \sqrt{\frac{m}{m+1}gH}. \tag{2.2}$$

Let $\frac{dt}{ds}$ be equal to 1. Then using equation (2.2), it follows that $\frac{dx}{ds} = \lambda_\pm$ and $\frac{dx}{dt} = \lambda_\pm$. Using these results we have that behaviors of H and u along the curve C_+ can be expressed through the system

$$\frac{d}{ds}\begin{pmatrix} H \\ u \end{pmatrix} = (\lambda_+ I - M)\begin{pmatrix} H \\ u \end{pmatrix}_x + \begin{pmatrix} 0 \\ -\alpha g \end{pmatrix}$$

and their behaviors along the curve C_- can be expressed through the system

$$\frac{d}{ds}\begin{pmatrix} H \\ u \end{pmatrix} = (\lambda_- I - M)\begin{pmatrix} H \\ u \end{pmatrix}_x + \begin{pmatrix} 0 \\ -\alpha g \end{pmatrix}$$

respectively. Let us introduce Riemann invariants (*DuChateau and Zachmann*, 2011) I_+ and I_- so that

$$M^T\begin{pmatrix} \frac{\partial I_\pm}{\partial H} \\ \frac{\partial I_\pm}{\partial u} \end{pmatrix} = \lambda_\pm \begin{pmatrix} \frac{\partial I_\pm}{\partial H} \\ \frac{\partial I_\pm}{\partial u} \end{pmatrix}, \text{ where } M^T \text{ is the transpose of } M. \tag{2.3}$$

Notice that both matrices $\lambda_+ I - M$ and $\lambda_- - M$ are singular. Since these matrices are singular, so are their transposes $\lambda_+ I - M^T$ and $\lambda_- I - M^T$. Then we can choose invariants $I_\pm$ so that one of their partial derivatives is constant. Using flexibility of this choice, let us pick the invariants I_+ and I_- so that

$$\frac{\partial I_\pm}{\partial u} = 1 \text{ and } \frac{\partial I_\pm}{\partial H} = \pm\sqrt{\frac{m+1}{m}\frac{g}{H}}. \tag{2.4}$$

Multiplying both sides of equation (2.1) by the row $\begin{pmatrix} \frac{\partial I_\pm}{\partial H} & \frac{\partial I_\pm}{\partial u} \end{pmatrix}$ on the left and applying (2.3) we obtain the following equality:

$$\frac{\partial I_\pm}{\partial t} + \lambda_\pm \frac{\partial I_\pm}{\partial x} = -\alpha g. \tag{2.5}$$

Introducing new variables J_+ and J_- so that

$$J_\pm = I_\pm + \alpha g t \tag{2.6}$$

we obtain the homogeneous system of equations

$$\frac{\partial J_\pm}{\partial t} + \lambda_\pm \frac{\partial J_\pm}{\partial x} = 0. \tag{2.7}$$

It will let us make the variables x and t dependent. Here it becomes obvious why we chose the bay to have a constant slope in the offshore direction, so that the depth grows linearly. If the change of the depth was nonlinear, the right part of the system (1.4) would depend on x and we would not be able to make this system homogeneous.

2.2 Change of variables

Now we will make the variables x and t dependent. Let us consider Jacobian matrices $\frac{\partial(J_+,\ J_-)}{\partial(x,\ t)}$ and $\frac{\partial(x,\ t)}{\partial(J_+,\ J_-)}$. Using equation (2.7) we have that

$$\frac{\partial(J_+,\ J_-)}{\partial(x,\ t)} = \begin{pmatrix} \frac{\partial J_+}{\partial x} & -\lambda_+ \frac{\partial J_+}{\partial x} \\ \frac{\partial J_-}{\partial x} & -\lambda_- \frac{\partial J_-}{\partial x} \end{pmatrix}.$$

Since, the Jacobian matrix $\frac{\partial(x,\ t)}{\partial(J_+,\ J_-)}$ is the inverse of the Jacobian matrix $\frac{\partial(J_+,\ J_-)}{\partial(x,\ t)}$ we have that

$$\frac{\partial x}{\partial J_\pm} = \pm\lambda_\mp \frac{\partial J_\mp}{\partial x} / det \left(\frac{\partial(J_+,\ J_-)}{\partial(x,\ t)} \right) \text{ and } \frac{\partial t}{\partial J_\pm} = \pm \frac{\partial J_\mp}{\partial x} / det \left(\frac{\partial(J_+,\ J_-)}{\partial(x,\ t)} \right).$$

Combining these results we obtain the following system of equations:

$$\frac{\partial x}{\partial J_\pm} = \lambda_\mp \frac{\partial t}{\partial J_\pm}. \tag{2.8}$$

So, the variables x and t which were initially independent variables, became dependent. At the same time the Riemann invariants J_+, J_- became independent. Let us denote

$$J = \frac{\partial(x,\ t)}{\partial(J_+,\ J_-)}. \tag{2.9}$$

The determinant of this matrix shows possibility of transformation of the system (2.7) from the variables $(x,\ t)$ to the variables $(J_+,\ J_-)$ and back. If, for example, the transformation from the variables $(x,\ t)$ to the variables $(J_+,\ J_-)$ becomes impossible or defective, it cannot be applied in these cases. We will return to this Jacobian $det(J)$ later, when we will consider applicability limits of this transformation.

Now, in order to be able to work further with the system (2.8) let us obtain formulas for the Riemann invariants J_+ and J_-. Recalling, how these invariants were defined through invariants $I_\pm$ (given in the system (2.6)) and recalling the definition of the invariants $I_\pm$ (given in the system (2.4)), we have that we can write the invariants J_+ and J_- as

$$\begin{aligned} J_\pm(u, H) &= \int \frac{\partial I_\pm(u,\ H)}{\partial u} du \pm \int_0^H \frac{\partial I_\pm(u,\ H)}{\partial H} dH + \alpha g t \\ &= I_\pm(u = 0, H = 0) + \int_0^u du' \pm \sqrt{\frac{m+1}{m} g} \int_0^H \frac{dH'}{\sqrt{H'}} + \alpha g t \\ &= I_\pm(u = 0, H = 0) + u \pm 2\sqrt{\frac{m+1}{m} g H} + \alpha g t. \end{aligned}$$

Now, let without loss of generality I_+ and I_- for the zero values of u and H be equal 0. (Assume, they do not equal zero for $u = 0$ and $H = 0$. Since u and H are the only arguments of these functions I_+ and I_-, I_+(u=0,H=0) and I_-(u=0,H=0) are just constants. Then we can slightly modify the choice of J_+ and J_- deducting these constants from them. Since we change J_+ and J_- only by constants, it does not affect on the transformation results). Then

$$J_\pm = u \pm 2\sqrt{\frac{m+1}{m} gH} + \alpha g t.$$

We have that the formulas for both invariants J_+ and J_- involve 3 variables. Let us for simplicity introduce new variables:

$$\lambda = \frac{J_+ + J_-}{2} \text{ and } \sigma = \frac{J_+ - J_-}{2}. \tag{2.10}$$

For λ and σ we have

$$\lambda = u + \alpha g t \text{ and } \sigma = 2\sqrt{\frac{m+1}{m} gH}. \tag{2.11}$$

Now we want to express the system of equations (2.8) in these new variables, replacing the Riemann invariants. First, from the system (2.10) the Jacobian matrix of transformation from the variables J_+ and J_- to the variables λ and σ is equal to

$$\frac{(\partial\lambda,\ \partial\sigma)}{(\partial J_+,\ \partial J_-)} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Computing the inverse of this matrix we have that the Jacobian matrix for the translation from λ, σ back to J_+ and J_- is

$$\frac{(\partial J_+,\ \partial J_-)}{(\partial\lambda,\ \partial\sigma)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Also, let for simplicity write the system (2.8) in the matrix form:

$$\begin{pmatrix} \frac{\partial x}{\partial J_+} \\ \frac{\partial x}{\partial J_-} \end{pmatrix} = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial J_+} \\ \frac{\partial t}{\partial J_-} \end{pmatrix}.$$

Combining these results together we have that in terms of variables λ and σ this system looks as

$$\begin{pmatrix} x_\lambda \\ x_\sigma \end{pmatrix} = \frac{(\partial J_+,\ \partial J_-)}{(\partial\lambda,\ \partial\sigma)} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \frac{(\partial\lambda,\ \partial\sigma)}{(\partial J_+,\ \partial J_-)} \begin{pmatrix} t_\lambda \\ t_\sigma \end{pmatrix}.$$

Using the values of Jacobian matrices used in the system and the formulas for λ_+ and λ_- (given in (2.2)) and multiplying these matrices we obtain the following representation for the system (2.8)

$$\begin{pmatrix} x_\lambda \\ x_\sigma \end{pmatrix} = \begin{pmatrix} u & -\sqrt{\frac{m}{m+1} gH} \\ -\sqrt{\frac{m}{m+1} gH} & u \end{pmatrix} \begin{pmatrix} t_\lambda \\ t_\sigma \end{pmatrix}. \tag{2.12}$$

Now, let us express the system (2.12) in terms of λ and σ. Using the results from the system (2.11) we obtain the following form of the system (2.12):

$$\begin{pmatrix} x_\lambda \\ x_\sigma \end{pmatrix} = \begin{pmatrix} u & -\sqrt{\frac{m}{m+1}gH} \\ -\sqrt{\frac{m}{m+1}gH} & u \end{pmatrix} \begin{pmatrix} t_\lambda \\ t_\sigma \end{pmatrix}.$$

Let us eliminate H from this system. From the formula for σ in the system (2.11), we have that

$$\sqrt{H} = \sqrt{\frac{m}{m+1}}\frac{\sigma}{2\sqrt{g}}.$$

Substituting this result into the system yields the system

$$\begin{pmatrix} x_\lambda \\ x_\sigma \end{pmatrix} = \begin{pmatrix} u & -\frac{m}{2(m+1)}\sigma \\ -\frac{m}{2(m+1)}\sigma & u \end{pmatrix} \begin{pmatrix} t_\lambda \\ t_\sigma \end{pmatrix}. \tag{2.13}$$

Next, to in order make the further simplification step we will write the system (2.13) as one partial differential equation. Differentiation of the first equation with respect to σ and differentiation of the second equation with respect to λ yields:

$$\begin{cases} x_{\lambda\sigma} = u_\sigma t_\lambda + u t_{\lambda\sigma} - \frac{m}{2(m+1)}(t_\sigma + \sigma t_{\sigma\sigma}), \\ x_{\sigma\lambda} = -\sigma\frac{m}{2(m+1)}t_{\lambda\lambda} + u_\lambda t_\sigma + u t_{\sigma\lambda}. \end{cases}$$

Assuming that all partial derivatives of second order of $x(\lambda,\sigma)$ and $t(\lambda,\sigma)$ are continuous, we have that $x_{\lambda\sigma} = x_{\sigma\lambda}$ and $t_{\lambda\sigma} = t_{\sigma\lambda}$ The equality $x_{\lambda\sigma} = x_{\sigma\lambda}$ gives us the equality of the rights sides of equations and so,

$$u_\sigma t_\lambda + u t_{\lambda\sigma} - \frac{m}{2(m+1)}(t_\sigma + \sigma t_{\sigma\sigma}) = -\sigma\frac{m}{2(m+1)}t_{\lambda\lambda} + u_\lambda t_\sigma + u t_{\sigma\lambda}.$$

Next, the equality $t_{\lambda\sigma} = t_{\sigma\lambda}$ lets us to cancel the terms $ut_{\lambda\sigma}$ and $ut_{\sigma\lambda}$, yielding equation

$$\frac{m\sigma}{2(m+1)}(t_{\lambda\lambda} - t_{\sigma\sigma}) + u_\sigma t_\lambda - u_\lambda t_\sigma - \frac{m}{2(m+1)}t_\sigma = 0.$$

Then the formula for λ from the system (2.11) gives that

$$t = \frac{\lambda - u}{\alpha g} \tag{2.14}$$

and consequently partial derivatives of t and u are related through the following formulas:

$$\begin{aligned} t_\lambda &= \tfrac{1-u_\lambda}{\alpha g}, & t_{\lambda\lambda} &= -\tfrac{u_{\lambda\lambda}}{\alpha g}, \\ t_\sigma &= -\tfrac{u_\sigma}{\alpha g}, \text{ and } & t_{\sigma\sigma} &= -\tfrac{u_{\sigma\sigma}}{\alpha g}. \end{aligned} \tag{2.15}$$

Plugging these results into the previous equation, and then dividing both parts by the coefficient of the term $u_{\lambda\lambda}$ we obtain the homogeneous partial differential equation of 1 dependent variable u which has a form

$$u_{\lambda\lambda} - u_{\sigma\sigma} - \frac{3m+2}{m\sigma}u_\sigma = 0. \tag{2.16}$$

Expressing partial derivatives of u through partial derivatives of t and plugging these formulas into equation (2.17) yields the similar homogeneous partial differential equation in terms of t.

$$t_{\lambda\lambda} - t_{\sigma\sigma} - \frac{3m+2}{m\sigma} t_\sigma = 0. \tag{2.17}$$

Now, we would like to express the physical coordinate x through λ and σ. With this purpose let us introduce the potential $\Phi(\lambda, \sigma)$ so that

$$\Phi_\sigma = u\sigma \tag{2.18}$$

and for the undisturbed water surface Φ equals 0. Then equation (2.16) is converted into equation

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{m+2}{m}\frac{\Phi_\sigma}{\sigma} = 0, \tag{2.19}$$

and u can be expressed in terms of Φ by the formula

$$u = \frac{\Phi_\sigma}{\sigma}, \tag{2.20}$$

and the formula for λ from (2.11) lets us to express the time t through Φ also. We will call equation (2.19) the linearized shallow water equation.

Next we will express the variable x through Φ. Let us consider the second equation in the system (2.13), which is

$$x_\sigma = -\frac{m}{2(m+1)}\sigma t_\lambda + u t_\sigma.$$

Let us express t_λ through Φ using the results from (2.15) and equation (2.20). Then

$$t_\lambda = \frac{1}{\alpha g}\left(1 - \frac{\Phi_{\sigma\lambda}}{\sigma}\right).$$

Let us express t_σ through u using the results (2.15). Assume, that Φ has continuous second derivatives, so that $\Phi_{\lambda\sigma} = \Phi_{\sigma\lambda}$. Together these results give us equation

$$\alpha g x_\sigma = \frac{m}{2(m+1)}(\Phi_{\sigma\lambda} - \sigma) - u u_\sigma.$$

Applying equations (2.11), (1.2) and applying the assumptions on the velocity u and the potential Φ (for an undisturbed water surface both equal 0) we get the following formulas

$$x = \frac{1}{2\alpha g}\left(\frac{m}{m+1}\left[\Phi_\lambda - \frac{\sigma^2}{2}\right] - u^2\right), \tag{2.21}$$

$$\eta(\lambda, \sigma) = \frac{1}{2g}\left[\frac{m}{m+1}\Phi_\lambda - u^2\right], \tag{2.22}$$

and

$$H = \frac{m}{m+1}\frac{\sigma^2}{4g}. \tag{2.23}$$

Let us write the formulas for all physical variables together. Combining the formulas (2.14), (2.20), (2.22), (2.23), (2.25) and (2.21) we have the parametric system of physical variables:

$$\begin{array}{lll} u = \frac{\Phi_\sigma}{\sigma}, & t = \frac{\lambda - u}{\alpha g}, & H = \frac{m}{m+1}\frac{\sigma^2}{4g}, \\ \eta = \frac{1}{2g}\left[\frac{m}{(m+1)}\Phi_\lambda - u^2\right], & x = \frac{1}{2\alpha g}\left(\frac{m}{m+1}\left[\Phi_\lambda - \frac{\sigma^2}{2}\right] - u^2\right), & \lambda,\, \sigma \geq 0. \end{array} \tag{2.24}$$

If the hodograph type transformation is invertible, these formulas give us the inverse transform from nonphysical variables λ and σ, back to physical variables u, H, x, t. The values of these physical variables form the solution to the system of shallow water equations (1.4). The next step is to determine conditions which provide existence, uniqueness and invertibility of the hodograph transformation, which we use to linearize the shallow water equation. It is clear that, if for some pair (λ_0, σ_0), the reverse transformation into physical space is undefined, then the solution Φ and physical formulas do not have any physical meaning at this point.

2.3 Boundary conditions, initial conditions and domain of Φ, the linearized shallow water equation

First, we would like to determine the domain for Φ, the range of values λ and σ where the solution Φ is defined. Let us recall the system (2.11) and equation (1.2). The formula for σ from (2.11), which is

$$\sigma = 2\sqrt{\frac{m+1}{m}gH}$$

together with the formula (2.11), which is

$$H = \eta - \alpha x$$

give us

$$\sigma = 2\sqrt{\frac{m+1}{m}g(\eta - \alpha x)},$$

where η is the vertical water displacement and α is the slope of the bay. Informally speaking, σ plays the role of the distance from the shore. Similarly, recalling the formula

$$\lambda = u + \alpha g t,$$

the variable λ, plays the role of the time. Since we assumed that at initial time instant ($t = 0$), the velocity u also equals 0, by the previous formula we have that the initial instant of time ($t = 0$) corresponds to $\lambda = 0$.

Next, since for every point of the flooded area $H > 0$, then for every such point $\sigma > 0$ and $\sigma = 0$ corresponds to the floating shoreline.

Assuming that the functions u, and H are continuous, we also have that if the transformation from the coordinates (x, t) to the coordinates (λ, σ) is unique, then for each fixed physical point x, λ monotonically increases with time t and σ monotonically increases with the distance from shore.

Then, if the transformation into the variables λ and σ is defined uniquely, then for each physical point in water, (x,t),

$$\lambda \geq 0 \text{ and } \sigma > 0$$

(and the values $\sigma = 0$ corresponds to the shoreline and σ converging ∞ corresponds to the direction offshore).

Remark: in fact the transformation can produce negative values of λ for some positive instants of time. However, since for unperturbed water surface positive instants of time correspond to the positive instants of λ and the velocity u is continuous, it would immediately mean that some positive value of t is mapped to $\lambda = 0$. Since $\lambda = 0$ corresponds to $t = 0$, it would mean the failure of the uniqueness of the transformation.

In practice, the failure of uniqueness of this transformation corresponds to the wave breaking. However, if the wave breaks, the system of shallow water equations (1.1) is no longer valid to describe this wave. Therefore, $\Phi(\lambda,\sigma)$, the solution of the linearized shallow water equation (2.19), has the physical meaning as long as $\lambda, \sigma \geq 0$. Then we need to look for a nontrivial solution Φ only for those pairs (λ,σ) where both λ and σ are nonnegative.

In order to find a nontrivial solution for Φ for nonnegative λ and σ, let us find all the conditions which the solution $\Phi(\lambda,\sigma)$ has to satisfy to have the physical meaning.

First, let us consider Φ for $\sigma = 0$. For $\sigma = 0$ Φ will describe the wave behavior at the shoreline. Assuming that the horizontal water displacement u is always finite and using the formula (2.18) we have that

$$\Phi_\sigma(\lambda,\sigma)|_{\sigma=0} = 0.$$

Assuming that Φ is continuous for positive σ, we have that for each λ, $\Phi(\lambda,\sigma)|_{\sigma=0}$ is finite. This yields the other condition for Φ:

$$|\Phi(\lambda,\sigma)|_{\sigma=0}| < +\infty.$$

Secondly, let us consider Φ for $\sigma \to +\infty$. As we have shown, direction corresponding to $\sigma \to +\infty$ is the offshore direction. Since in the offshore direction water depth increases, every wave moving offshore will be dropping. Then water located infinitely far offshore remains undisturbed. Since Φ was introduced so that it equals 0 for undisturbed water surface, we have that

$$\lim_{\sigma \to +\infty} \Phi(\lambda,\sigma) = 0.$$

Since σ corresponds to the distance we will call the conditions for the solution Φ for $\sigma = 0, +\infty$ the boundary conditions.

Third, let us consider Φ for $\lambda = 0$. Since the variable λ corresponds to the time, we will call these conditions the initial conditions.

First, let us consider the partial derivative of Φ with respect to λ. Solving (2.22) with respect to λ we have that,

$$\Phi_\lambda(\lambda,\sigma) = \frac{m+1}{m}(2g\eta + u^2).$$

Plugging $\lambda = 0$ gives us

$$\Phi_\lambda(\lambda,\sigma)|_{\lambda=0} = \frac{m+1}{m}(2g\eta(\lambda,\sigma)|_{\lambda=0} + u^2|_{\lambda=0}(\lambda,\sigma)).$$

Since $\lambda = 0$ corresponds to the initial time instant and for the initial time instant u equals zero, we have that

$$\Phi_\lambda(\lambda,\sigma)|_{\lambda=0} = 2g\frac{m+1}{m}\eta(\lambda,\sigma)|_{\lambda=0}.$$

Second, let us consider Φ for $\lambda = 0$. Substituting the $\lambda = 0$ into (2.18) gives us

$$\Phi_\sigma(\lambda,\sigma)|_{\lambda=0} = \sigma \cdot u|_{\lambda=0}.$$

Since $u|_{\lambda=0} = 0$,

$$\Phi_\sigma(\lambda,\sigma)|_{\lambda=0} = 0.$$

Assuming that Φ is differentiable with respect to σ for every $\sigma > 0$ and continuous at $\lambda = 0$, we have that

$$(\Phi(\lambda,\sigma)|_{\lambda=0})_\sigma = 0,$$

which implies that $\Phi(\lambda,\sigma)|_{\lambda=0}$ is just a constant. Since Φ was introduced to be 0 for the calm water surface,

$$\Phi(\lambda,\sigma)|_{\lambda=0} = 0.$$

Combining all the results together we have that $\Phi(\lambda,\sigma)$, the solution of the linearized shallow water equation (2.19), must satisfy the following conditions:

1) It can be nontrivial only for

$$\lambda, \sigma \geq 0, \tag{2.25}$$

2) initial conditions

$$\Phi(\lambda,\sigma)|_{\lambda=0} = 0, \quad \Phi_\lambda(\lambda,\sigma)|_{\lambda=0} = 2\tfrac{m+1}{m}g\eta|_{\lambda=0}, \tag{2.26}$$

3) boundary conditions

$$|\Phi(\lambda,\sigma)\;|_{\sigma=0}\;| < \infty, \quad \Phi_\sigma(\lambda,\sigma)|_{\sigma=0} = 0, \text{ and } \quad \lim_{\sigma\to+\infty}\Phi(\lambda,\sigma) = 0. \tag{2.27}$$

2.4 The limits of applicability of the hodograph transformation.

Let us recall the formula for λ in (2.11):

$$\lambda = u + \alpha g t.$$

It can be possible that this formula can map two 2 different instants of time to the same value of λ. If that happens, all the hodograph transformation from physical coordinates to λ and σ, becomes invalid. As a result, the uniquely defined transformation back to physical coordinates does not exist.

Also, it seems possible that equation (2.14) can map 2 different points of (λ, σ) space into the same physical time, which would mean that the inverse transformation - from the variables (λ, σ) to the variables (x, t) does not exist. Therefore, we need to use this transformation carefully. What we would like to do is to be able to find and describe the conditions when the transformation fails.

Recall first, that in the process of constructing the hodograph transformation, we introduced the Riemann invariants J_+ and J_- and expressed through them the system of shallow water equations (1.4). Second, it is easy to check, there is a one-to one correspondence between the invariants J_+, J_- and the variables λ, σ. Hence, the translation from the variables J_+, J_- to the variables λ, σ is always defined uniquely and invertible.

Therefore all the hodograph transformation depends only on the translation from physical variables (x, t) to the system Riemann invariants (J_+, J_-) and back. Let us consider the Jacobian matrix J (2.9), which is exactly determines this translation. As one can see, this translation is well-defined and is invertible as long as the Jacobian $det(J)$ neither blows up nor vanishes.

Let us consider the matrix for this Jacobian. By equation (2.9)

$$J = \begin{pmatrix} \frac{\partial x}{\partial J_+} & \frac{\partial x}{\partial J_-} \\ \frac{\partial t}{\partial J_+} & \frac{\partial t}{\partial J_-} \end{pmatrix}.$$

Let us simplify the formula for this matrix. For simplicity, let us eliminate the differentiable variable x. The formulas (2.7) yield that

$$J = \begin{pmatrix} \lambda_- \frac{\partial t}{\partial J_+} & \lambda_+ \frac{\partial t}{\partial J_-} \\ \frac{\partial t}{\partial J_+} & \frac{\partial t}{\partial J_-} \end{pmatrix}$$

and then the Jacobian can be expressed as

$$det(J) = (\lambda_- - \lambda_+) \frac{\partial t}{\partial J_+} \frac{\partial t}{\partial J_-},$$

where by the system (2.2),

$$\lambda_+ = u + \sqrt{\frac{m}{m+1} gH} \text{ and } \lambda_- = u - \sqrt{\frac{m}{m+1} gH}.$$

The formulas for λ_+ and λ_- give us

$$det(J) = -2\sqrt{\frac{m}{m+1} gH} \frac{\partial t}{\partial J_+} \frac{\partial t}{\partial J_-}.$$

Next, we would like to express the determinant in terms of familiar variables λ and σ.

Applying the chain rule yields that

$$\frac{\partial t}{\partial J_\pm} = t_\lambda \frac{\partial \lambda}{\partial J_\pm} + t_\sigma \frac{\partial \sigma}{\partial J_\pm},$$

and (2.10) implies

$$\frac{\partial t}{\partial J_\pm} = \frac{1}{2} \left[t_\lambda \pm t_\sigma \right].$$

Combining these results together, we have that

$$det(J) = \frac{1}{2}\sqrt{\frac{m}{m+1}gH}\left[t_\sigma^2 - t_\lambda^2\right].$$

Expressing H through the nonphysical variable σ by (2.23) we have

$$det(J) = \frac{m}{m+1}\frac{\sigma}{(2\alpha g)^2}\left[t_\sigma^2 - t_\lambda^2\right].$$

Notice that in the process of computing the physical variables through λ and σ we first compute u (after the computation of Φ_σ and using the formula (2.20)) and only after that, equation (2.14) lets us compute the variable t.

Then, it would be more convenient to express the Jacobian $det(J)$ in terms of the variable u. Using the relation between the partial derivatives of t and u from the system (2.15) and substituting the results for the derivatives of t in (2.15) we obtain the formula

$$det(J) = \frac{m}{m+1}\frac{\sigma}{(2\alpha g)^2}\left[u_\sigma^2 - (1-u_\lambda)^2\right]. \tag{2.28}$$

As we showed, we can use this hodograph transformation only if the determinant of J does not blow up and does not change its sign - at least for initial conditions such as the initial water displacement. However, in the practice the determinant should not blow-up. The blow up of the Jacobian $det(J)$ for any point located on a finite distance from offshore (so that σ is finite) would immediately imply either blow-up of function of acceleration, or discontinuity of velocity of a water displacement as function of a coordinate. In the first case by the second Newton's Law, it would mean that the physical force grows up to infinity. But since every physical force is always finite, it is impossible. Second, the blow-up with respect to the function of distance is also impossible in the model, since water is incompressible.

Hence, the only possible issue which can exist for the Jacobian $det(J)$ is that it vanishes at some points. First, the determinant vanishes if $\sigma \to 0$. Since it is a boundary point of the flooded area, we do not need to worry about this case. However, it also vanishes if the absolute values $|1 - u_\lambda|$ and $|u_\sigma|$ get equal. If they get equal, then translation from the Riemann invariants back to the physical coordinates no longer exists. On the other hand this fact gives us the opportunity to check whether we can translate the system back to the physical coordinates or not. Also, it allows us to determine, whether the wave breaks for given physical coordinates or not. If the determinant vanishes, in practice it corresponds to immediate break of the wave. The immediate break of the wave implies the system of shallow water equations is invalid for describing the behavior of such process.

Suppose, that the hodograph transform is valid for the initial conditions but for some positive values of λ the Jacobian crosses the zero-level. Then as one can see we can compute the physical characteristics only for that area of $\lambda \times \sigma$ plane if that area satisfies the following conditions:

1) it satisfies the restrictions for λ and σ given in (2.25): $\lambda, \sigma \geq 0$,

2) it contains all points $(0, \sigma)$,

3) the Jacobian on this area $det(J)$ is either nonnegative or non-positive,

4) the interior of this area is simply connected.

Chapter 3

Solution of the linearized shallow water equation (the equation (2.19))

In the previous sections we discussed the transform from the nonlinear system of shallow water equations (1.4) with initial conditions $u|_{t=0} = 0$ and given $H|_{t=0}$ to the partial differential equation (2.19) which has the boundary conditions (2.27) and initial conditions (2.26) for $m \in (0, \infty]$.

Since this is a partial differential equation, would like to simplify it. Let us try to convert this equation to the ordinary differential equation. Since the solution Φ can be nontrivial for nonnegative values of λ, and no term of this equation contains λ, we decided to apply the Laplace transformation to this equation with respect to the variable λ.

3.1 Laplace transformation

Let $\hat{\Phi}(\sigma, s) = \mathcal{L}(\Phi(\lambda, \sigma))$ be the image of $\Phi(\lambda, \sigma)$ under the Laplace transform. Let us assume that Φ has appropriate properties so that we can interchange the orders of differentiation with respect to σ and Laplace transformation with respect to λ. Then under the Laplace transform equation (2.19) is converted into equation

$$\hat{\Phi}_{\sigma\sigma} + \frac{m+2}{m}\frac{\hat{\Phi}_\sigma}{\sigma} - s^2\hat{\Phi} + s\Phi|_{\lambda=0} + \Phi_\lambda|_{\lambda=0} = 0.$$

Let us return to the initial conditions for (2.19) given in (2.26). Recall that $\Phi|_{\lambda=0} = 0$ and notice that $\Phi_\lambda|_{\lambda=0}$ is a function which depends only on the variable σ. Let us denote it as $R(\sigma)$. Then the given differential equation becomes

$$\hat{\Phi}_{\sigma\sigma} + \frac{m+2}{m}\frac{\hat{\Phi}_\sigma}{\sigma} - s^2\hat{\Phi} = -R, \tag{3.1}$$

where R does not depend on the parameter s. (The choice of a sign for the function R has been done on purpose. Later you will see, why it has been done this way). So, the linear partial differential equation (2.19) has been converted into the linear ordinary nonhomogeneous equation (3.1).

3.2 Solving the transformed equation

Now we will solve this ordinary differential equation. Let us solve it by the method of variation of parameters (*Zill*, 2000). Then, first, let us solve the homogeneous equation

$$\hat{\Phi}_{\sigma\sigma} + \frac{m+2}{m}\frac{\hat{\Phi}_\sigma}{\sigma} - s^2\hat{\Phi} = 0. \tag{3.2}$$

Let $\beta = \frac{1}{m}$. Then equation (3.2) becomes

$$\hat{\Phi}_{\sigma\sigma} + (1+2\beta)\frac{\hat{\Phi}_\sigma}{\sigma} - s^2\hat{\Phi} = 0.$$

In order to simplify this equation let us make the substitution $\hat{\Phi} = \sigma^{-\beta}Y$. It follows that

$$\hat{\Phi}_\sigma = \sigma^{-\beta}(Y_\sigma - \beta\sigma^{-1}Y)$$

and

$$\hat{\Phi}_{\sigma\sigma} = \sigma^{-\beta}(Y_{\sigma\sigma} - 2\beta\sigma^{-1}Y_\sigma + \beta(\beta+1)\sigma^{-2}Y).$$

Substituting these formulas into equation (3.2) and combining the homogeneous terms we obtain equation

$$\sigma^{-\beta}\left[Y_{\sigma\sigma} + \frac{1}{\sigma}Y_\sigma - \left(\frac{\beta^2}{\sigma^2} + s^2\right)Y\right] = 0.$$

Now we need to simplify it further. Multiplying both parts of this equation by $\sigma^{2+\beta}$ yields equation

$$\sigma^2 Y_{\sigma\sigma} + \sigma Y_\sigma - (s^2\sigma^2 + \beta^2)Y = 0.$$

Let $\tau = \sigma s$. Then, since s and σ are independent variables, using the chain rule we have that $Y_\sigma = Y_\tau \cdot \tau_\sigma = sY_\tau$ and similarly, $Y_{\sigma\sigma} = s^2 Y_{\tau\tau}$. Substituting these results into previous equation we obtain

$$\sigma^2 s^2 Y_{\tau\tau} + \sigma s Y_\tau - (\sigma^2 s^2 + \beta^2)Y = 0,$$

and the substitution of σ by τ gives us the result

$$\tau^2 Y_{\tau\tau} + \tau Y_\tau + (\tau^2 - \beta^2)Y = 0,$$

which is known as Modified Bessel equation ((*Abramowitz and Stegun*, 1965) (9.6.1)) . The linearly independent solutions of this equation are Modified Bessel functions $I_\beta(\tau)$ (*Abramowitz and Stegun*, 1965)(9.6.3) and $K_\beta(\tau)$ (*Abramowitz and Stegun*, 1965)(9.6.2). Let

$$Y_1 = I_\beta(\tau) \text{ and } Y_2 = K_\beta(\tau),$$

be the solutions of the Modified Bessel equation. Substituting τ back with σ we have that $Y_1 = I_\beta(\sigma s)$ and $Y_2 = K_\beta(\sigma s)$. Let S_1 and S_2 be linearly independent solutions of equation (3.2) expressed through Y_1 and Y_2 respectively. Then going back from the variable Y to the variable $\hat{\Phi}$ we have that

$$S_1(\sigma, s) = \sigma^{-\beta} I_\beta(\sigma s) \text{ and } S_2(\sigma, s) = \sigma^{-\beta} K_\beta(\sigma s).$$

Hence, the function

$$\hat{\Phi}_h = c_1\sigma^{-\beta} S_1(\sigma, s) + c_2\sigma^{-\beta} S_2(\sigma, s)$$

is the solution of the homogeneous equation (3.2), where c_1 and c_2 are constant with respect to σ (but might depend on s) and $\beta = \frac{1}{m}$.

Let $\hat{\Phi}(\lambda, \sigma)$ be the solution of equation (3.1) for $s > 0$ and $\sigma > 0$. Then using variation parameters we have that

$$\hat{\Phi}(\sigma, s) = A(\sigma, s)S_1(\sigma, s) + B(\sigma, s)S_2(\sigma, s),$$

where

$$A_\sigma(\sigma, s) = \frac{S_2(\sigma,s)R(\sigma)}{W(S_1(\sigma), S_2(\sigma))}, \quad B_\sigma(\sigma, s) = -\frac{S_2(\sigma,s)R(\sigma)}{W(S_1(\sigma), S_2(\sigma))}, \text{ and}$$

$$W(S_1(\sigma,s),S_2(\sigma,s)) = det\begin{pmatrix} S_1(\sigma,s) & S_2(\sigma,s) \\ (S_1)_\sigma(\sigma,s) & (S_2)_\sigma(\sigma,s)\end{pmatrix}$$

is the Wronskian of the functions S_1 and S_2 differentiated with respect to σ,
and

$$R(\sigma) = \Phi_\lambda(\lambda,\sigma)|_{\lambda=0}.$$

Let us find the Wronskian. Differentiating of the functions S_1 and S_2 with respect to σ through chain rule and combining the homogenous terms we have that

$$W(S_1(\sigma,s),S_2(\sigma,s)) = \sigma^{-2\beta} sW(I_\beta(\sigma s),\, K_\beta(\sigma s)),$$

where $W(I_\beta(\sigma s), K_\beta(\sigma s))$ is the Wronskian of the functions $I_\beta(\sigma s)$ and $K_\beta(\sigma s)$ differentiated with respect to σ. Substituting σs by τ and using the formula

$$W(I_\beta(\tau),K_\beta(\tau)) = -\frac{1}{\tau}$$

((*Abramowitz and Stegun*, 1965) (9.6.15)), we have that

$$W(S_1(\sigma,s),S_2(\sigma,s)) = -\sigma^{-(1+2\beta)}.$$

Now let us express the solution $\hat{\Phi}$ in terms of Modified Bessel functions. As one can easy check,

$$\hat{\Phi}(\sigma,s) = A(\sigma,s)\sigma^{-\beta}I_\beta(\sigma,s) + B(\sigma,s)\sigma^{-\beta}K_\beta(\sigma s), \tag{3.3}$$

where

$$A_\sigma(\sigma,s) = -\sigma^{1+\beta}K_\beta(\sigma s)R(\sigma), \text{ and } \quad B_\sigma(\sigma,s) = \sigma^{1+\beta}I_\beta(\sigma s)R(\sigma).$$

Now we need to find the functions A and B up to the constants. In order to be able to do it, we need to obtain the boundary conditions for $\hat{\Phi}$. We will derive them from boundary conditions for its preimage Φ, given in (2.27). Recall that $\Phi|_{\sigma=0}$ is finite, $\Phi_\sigma|_{\sigma=0} = 0$, and Φ converges to 0 for $\sigma \to +\infty$. This immediately gives us the boundary conditions that for any $s > 0$,

$$|\hat{\Phi}(\sigma,s)|_{\sigma=0}| < \infty \;,\; \hat{\Phi}(\sigma,s)_\sigma|_{\sigma=0} = 0 \text{ and } \lim_{\sigma\to\infty}\hat{\Phi}(\sigma,s) = 0;\; s > 0, \tag{3.4}$$

Now, using these conditions, let us determine the constants A and B. First, let us denote the additive terms of the solution $\hat{\Phi}$. Let

$$\hat{\Phi}_1 = A(\sigma,s)\sigma^{-\beta}I_\beta(\sigma s) \text{ and } \hat{\Phi}_2 = B(\sigma,s)\sigma^{-\beta}K_\beta(\sigma s),$$

so that $\hat{\Phi}$ can be represented as a sum:

$$\hat{\Phi} = \hat{\Phi}_1 + \hat{\Phi}_2.$$

First, let us consider $\hat{\Phi}$ (given in equation (3.3)) for σ converging to 0.

Since $\hat{\Phi}(\sigma, s)$ is finite for $\sigma = 0$, both additive terms of this solution, Φ_1 and Φ_2 must be finite. In particular, Φ_2 must be also finite. However, by the asymptotic formula for the function K (*Abramowitz and Stegun*, 1965)(9.6.9) (see also the Appendix A) it follows that for any $\beta \geq 0$, and $s > 0$, $K_\beta(\sigma s)$ converges to $+\infty$. Then $B(\sigma, s)$ must converge to 0. Considering its derivative B_σ, we have that the only possible choice for B is if

$$B(\sigma, s) = \int_0^\sigma \omega^{1+\beta} I_\beta(\omega s) R(\omega) d\omega.$$

Next, let us consider the formula (3.3) for σ converging to ∞. Since by equation (3.4) $\hat{\Phi}$ converges to 0, both additive terms of this solution $\hat{\Phi}_1$ and $\hat{\Phi}_2$ should be finite. In particular $\hat{\Phi}_1$ is finite. Let us consider its formula and in particular its factor $\sigma^{-\beta} I_\beta(\sigma s)$. However, by the asymptotic formula (*Abramowitz and Stegun*, 1965)(9.7.1) (see also the Appendix A) for any $x \to +\infty$ and any β $I_\beta(x)$ is asymptotically close to $\frac{e^x}{\sqrt{2\pi x}}$. So, for any $s > 0$ $I_\beta(\sigma s)$ converges to ∞. Since the exponential function grows faster than any power function σ^β, $\beta > 0$, the factor $\sigma^{-\beta} I_\beta(\sigma s)$ also converges to $+\infty$ for $\sigma \to +\infty$. This implies that for any $s > 0$ $A(\sigma, s)$ must converge to 0. Analyzing the formula for A_σ as one can see, the only possible choice for A is if

$$A(\sigma, s) = \int_\sigma^\infty \omega^{1+\beta} K_\beta(\omega s) R(\omega) d\omega.$$

Combining these results and writing them in the form of the Green's function (*Rybkin*, 2012), we have that at least for $\sigma, s > 0$ the solution $\hat{\Phi}$ should be

$$\hat{\Phi}(\sigma, s) = \int_0^\infty G(\sigma, \omega, s) R(\omega) d\omega \tag{3.5}$$

where

$$G(\sigma, \omega, s) = \sigma^{-\beta} \omega^{\beta+1} \begin{cases} I_\beta(\omega s) K_\beta(\sigma s) \text{ if } \omega \leq \sigma, \\ I_\beta(\sigma s) K_\beta(\omega s) \text{ if } \omega \geq \sigma, \end{cases}$$
$$R(\omega) = \Phi_\lambda|_{\lambda=0}(\lambda, \omega).$$

For future simplification let us rewrite the function $R(\sigma)$ in physical coordinates. By the system (2.26)

$$\Phi_\lambda(\lambda, \omega)|_{\lambda=0} = 2\frac{m+1}{m} g\eta|_{\lambda=0}(\lambda, \sigma),$$

and then

$$R(\sigma) = 2\frac{m+1}{m} g\eta|_{\lambda=0}(\lambda, \sigma). \tag{3.6}$$

The detailed asymptotical analysis of the solution (3.5) for σ converging both to 0 and ∞ which uses the asymptotic behavior of the Modified Bessel functions (*Abramowitz and Stegun*, 1965)(9.8.7-9, 9.7.1-2) (and see Appendix A) shows that if $R(\omega)$ (which as we see linearly depends on the initial vertical water displacement) is bounded and decays sufficiently quickly for $\sigma \to \infty$, then the solution (3.5) satisfies the boundary conditions (3.4). If this decay is not fast enough then it implies that the vertical water displacement carries infinite amount of energy. If it happens such a problem does not make physical sense.

3.3 Inverse Laplace transformation

Recall that we are looking for the solution of the linearized shallow water equation, equation (2.19). Let Φ be the such solution. In the previous chapter we have found an explicit formula for the image of Φ under Laplace transform denoted as $\hat{\Phi}$ and given in equation (3.5). Then

$$\Phi(\lambda, \sigma) = \mathcal{L}^{-1}(\hat{\Phi}(\sigma, s)).$$

and in order to find Φ, we need to compute the preimage of $\hat{\Phi}$ under the Laplace transform. Let us recall the formula for $\hat{\Phi}$. By equations (3.5), and (3.6) for $\sigma > 0$ and $s > 0$,

$$\hat{\Phi}(\sigma, s) = \int_0^\infty G(\sigma, \omega, s) R(\omega) d\omega,$$

where

$$G(\sigma, \omega, s) = \sigma^{-\beta}\omega^{\beta+1} \begin{cases} I_\beta(\omega s) K_\beta(\sigma s) \text{ if } \omega \leq \sigma, \\ I_\beta(\sigma s) K_\beta(\omega s) \text{ if } \omega \geq \sigma, \end{cases}$$

$$R(\omega) = 2\frac{m+1}{m} g\eta|_{\lambda=0}(\lambda, \sigma).$$

Then

$$\Phi(\lambda, \sigma) = \mathcal{L}^{-1}\left[\int_0^\infty G(\sigma, \omega, s) R(\omega) d\omega\right].$$

Using the property of linearity of the transformation and assuming the function $G(\sigma, \omega, s)$ sufficiently well behaves so that we can interchange intergration and taking the inverse of the transform without change of the result, we have that

$$\Phi(\lambda, \sigma) = \left[\int_0^\infty \mathcal{L}^{-1}\left(G(\sigma, \omega, s)\right) R(\omega) d\omega\right].$$

So, we need to find the preimage of $G(\sigma, \omega, s)$ under the Laplace transform. Notice that the Green's function G has a factor $\sigma^{-\beta}\omega^{1+\beta}$, which does not depend on s. The only factors which depend on s, are the Modified Bessel functions. Let us set

$$\overline{G}(\sigma, \omega, s) = \begin{cases} I_\beta(\omega s) K_\beta(\sigma s) & \text{if } \omega \leq \sigma, \\ I_\beta(\sigma s) K_\beta(\omega s) & \text{if } \omega \geq \sigma. \end{cases} \tag{3.7}$$

Then

$$G(\sigma, \omega, s) = \sigma^{-\beta}\omega^{1+\beta}\overline{G}(\sigma, \omega, s),$$

and by linearity of the Laplace transform,

$$\mathcal{L}^{-1}\left(G(\sigma, \omega, s)\right) = \sigma^{-\beta}\omega^{1+\beta}\mathcal{L}^{-1}\left(\overline{G}(\sigma, \omega, s)\right).$$

Let, without loss of generality fix $\omega \leq \sigma$ and so

$$\overline{G}(\sigma, \omega, s) = I_\beta(\omega s) K_\beta(\sigma s)$$

(otherwise, we just need to interchange σ with ω). Then

$$\mathcal{L}^{-1}\left(\overline{G}(\sigma,\omega,s)\right)=\mathcal{L}^{-1}\left(I_\beta(\omega s)K_\beta(\sigma s)\right)$$

and so, the problem of computation of the preimage of $\hat{\Phi}(\sigma,s)$ under Laplace transform has been converted into the problem of computation of the preimage of the product of two Modified Bessel functions.

First, let us investigate these functions more and obtain some of their properties.

By the formulas *Abramowitz and Stegun* (1965)(9.6.20) for any $x>0$,

$$I_\beta(x)=\frac{1}{\pi}\int_0^\pi \exp(x\cos\theta)\cos(\beta\theta)d\theta-\frac{\sin(\beta\pi)}{\pi}\int_0^\infty \exp(-x\cosh t-\beta t)dt \tag{3.8}$$

and by the formula *Abramowitz and Stegun* (1965)(9.6.24)

$$K_\beta(x)=\int_0^\infty \exp(-x\cosh t)\cosh(\beta t)dt. \tag{3.9}$$

In particular, for any positive τ and s,

$$I_\beta(\tau s)=\frac{1}{\pi}\int_0^\pi \exp(\tau s\cos\theta)\cos(\beta\theta)d\theta-\frac{\sin(\beta\pi)}{\pi}\int_0^\infty \exp(-\tau s\cosh t-\beta t)dt \tag{3.10}$$

and

$$K_\beta(\tau s)=\int_0^\infty \exp(-\tau s\cosh t)\cosh(\beta t)dt. \tag{3.11}$$

Notice that both Modified Bessel functions have exponents in their formulas and the parameter s present is only in the exponential of each exponent. Then in each exponent we would like to introduce a new variable μ so that the exponent of each exponential would simplify to $-\mu s$.

Let us first consider $I_\beta(\tau s)$. The formula for this function has two integrals. Making the substitutions $\mu=-\tau(\cos\theta)$ for the first one and $\mu=\tau\cosh t$ for the other we obtain that

$$I_\beta(\tau s)=\frac{1}{\pi}\int_{-\tau}^{\tau}\frac{\cos(\beta\arccos(-\mu/\tau))}{\sqrt{\tau^2-\mu^2}}e^{-\mu s}d\mu-\frac{\sin(\beta\pi)}{\pi}\int_\tau^\infty\frac{\exp(-\beta\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}e^{-\mu s}d\mu.$$

Let us for simplicity introduce the functions

$$i(\mu,\tau)=\frac{1}{\pi}\cdot\frac{\cos(\beta\arccos(-\mu/\tau))}{\sqrt{\tau^2-\mu^2}}\cdot\mathcal{U}(\tau-|\mu|) \tag{3.12}$$

and

$$j(\mu,\tau)=\frac{\sin(\pi\beta)}{\pi}\cdot\frac{\exp(-\beta\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}\mathcal{U}(\mu-\tau), \tag{3.13}$$

where $\mu\in(-\infty,\infty)$, and $\tau>0$. Then

$$I_\beta(\tau s)=\int_{-\tau}^\infty i(\mu,\tau)e^{-\mu s}d\mu-\int_0^\infty j(\mu,\tau)e^{-\mu s}d\mu.$$

Clearly, the first integral term of $I_\beta(\tau s)$ is not invertible under the Laplace transform. Then all the function $I_\beta(\tau s)$ is not invertible. However, it still can be represented in terms of the Laplace transform.

Let

$$I_\beta^1(\tau s) = \int_{-\tau}^{\infty} i(\mu,\tau)e^{-\mu s}d\mu$$

and let

$$I_\beta^2(\tau s) = \int_{0}^{\infty} j(\mu,\tau)e^{-\mu s}d\mu.$$

Then

$$I_\beta(\tau s) = I_\beta^1(\tau s) - I_\beta^2(\tau s).$$

Let us consider the function $I_\beta^1(\tau s)$. Let us make the change of variable in its formula substituting $\mu' = \mu + \tau$. Then

$$\begin{aligned} I_\beta^1(\tau s) &= \int_{-\tau}^{\infty} i(\mu,\tau)e^{-\mu s}d\mu \\ &= e^{\tau s}\cdot \int_0^{\infty} i(\mu'-\tau,\tau)e^{-\mu' s}d\mu' \\ &= e^{\tau s}\cdot \int_0^{\infty} i(\mu-\tau,\tau)e^{-\mu s}d\mu \\ &= e^{\tau s}\cdot \mathcal{L}\left[i(\mu-\tau,\tau)\right]. \end{aligned}$$

Therefore,

$$I_\beta^1(\tau s) = e^{\tau s}\cdot \mathcal{L}\left[i(\mu-\tau,\tau)\right]. \tag{3.14}$$

Let us consider the function $I_\beta^2(\tau s)$. Clearly,

$$I_\beta^2(\tau s) = \mathcal{L}\left[j(\mu,\tau)\right]. \tag{3.15}$$

Combining these results together we obtain

$$I_\beta(\tau s) = e^{\tau s}\cdot \mathcal{L}\left[i(\mu-\tau,\tau)\right] - \mathcal{L}\left[j(\mu,\tau)\right]. \tag{3.16}$$

Let us consider the function $K_\beta(\tau s)$, given in the formula (3.11). In the similar way, substituting into the formula $\mu = \tau\cosh t$ we have that

$$K_\beta(\tau s) = \int_\tau^\infty \frac{\cosh(\beta\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}e^{-\mu s}d\mu.$$

Let

$$k(\mu,\tau) = \frac{\cosh(\beta\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}\mathcal{U}(\mu-\tau). \tag{3.17}$$

Then

$$K_\beta(\tau s) = \int_0^\infty k(\mu,\tau)e^{-\mu s}d\mu = \mathcal{L}\left[k(\mu,\tau)\right]. \tag{3.18}$$

Let us return to the computation of the preimage of the function $\overline{G}(\sigma,\omega,s)$ under Laplace transform. Recall that without loss of generality we fixed $\omega \le \sigma$ and so,

$$\overline{G}(\sigma,\omega,s) = I_\beta(\omega s)K_\beta(\sigma s).$$

By the formulas (3.16) and (3.18)

$$\begin{aligned} I_\beta(\omega s)K_\beta(\sigma s) &= I^1_\beta(\omega s)\cdot K_\beta(\sigma s) - I^2_\beta(\omega s)\cdot K_\beta(\sigma s) \\ &= e^{\omega s}\mathcal{L}\left[i(\lambda-\omega,\omega)\right]\cdot\mathcal{L}\left[k(\lambda,\sigma)\right] - \mathcal{L}\left[j(\lambda,\omega)\right]\cdot\mathcal{L}\left[k(\lambda,\sigma)\right]. \end{aligned}$$

Let us consider $I^1_\beta(\omega s)\cdot K_\beta(\sigma s)$. Then

$$I^1_\beta(\omega s)\cdot K_\beta(\sigma s) = e^{\omega s}\mathcal{L}\left[i(\lambda-\omega,\omega)\right]\cdot\mathcal{L}\left[k(\lambda,\sigma)\right] = \mathcal{L}\left[i(\lambda-\omega,\omega)\right]\cdot\left(e^{\omega s}\mathcal{L}\left[k(\lambda,\sigma)\right]\right).$$

In particular, consider $e^{\omega s}\mathcal{L}\left[k(\lambda,\sigma)\right]$:

$$\begin{aligned} e^{\omega s}\mathcal{L}\left[k(\lambda,\sigma)\right] &= e^{\omega s}K_\beta(\sigma s) \\ &= e^{\omega s}\int_0^\infty k(\lambda,\sigma)e^{-\lambda s}d\lambda \\ &= \int_0^\infty k(\lambda,\sigma)e^{-(\lambda-\omega)s}d\lambda. \end{aligned}$$

Let us substitute $\lambda' = \lambda - \omega$. Then

$$e^{\omega s}\mathcal{L}\left[k(\lambda,\sigma)\right] = \int_{-\omega}^\infty k(\lambda'+\omega,\sigma)e^{-\lambda' s}d\lambda' = \int_{-\omega}^\infty k(\lambda+\omega,\sigma)e^{-\lambda s}d\lambda.$$

Notice that by formula (3.17) if $k(\lambda+\omega,\sigma)$, then $\lambda+\omega-\sigma>0$. Then $\lambda>\sigma-\omega$ Since $\omega\le\sigma$, $\lambda\ge 0$ and then we can write

$$e^{\omega s}\mathcal{L}\left[k(\lambda,\sigma)\right] = \int_0^\infty k(\lambda+\omega,\sigma)e^{-\lambda s}d\lambda = \mathcal{L}\left[k(\lambda+\omega,\sigma)\right].$$

Combining all these results gives us that for $\omega\le\sigma$

$$I_\beta(\omega s)K_\beta(\sigma s) = \mathcal{L}\left[i(\lambda-\omega,\sigma)\right]\cdot\mathcal{L}\left[k(\lambda+\omega,\sigma)\right] - \mathcal{L}\left[j(\lambda,\omega)\right]\cdot\mathcal{L}\left[k(\lambda,\sigma)\right].$$

Similarly for $\omega\ge\sigma$,

$$I_\beta(\sigma s)K_\beta(\omega s) = \mathcal{L}\left[i(\lambda-\sigma,\omega)\right]\cdot\mathcal{L}\left[k(\lambda+\sigma,\omega)\right] - \mathcal{L}\left[j(\lambda,\sigma)\right]\cdot\mathcal{L}\left[k(\lambda,\omega)\right].$$

Hence,

$$\overline{G}(\sigma,\omega,s) = \begin{cases} \mathcal{L}\left[i(\lambda-\omega,\omega)\right]\cdot\mathcal{L}\left[k(\lambda+\omega,\sigma)\right] - \mathcal{L}\left[j(\lambda,\omega)\right]\cdot\mathcal{L}\left[k(\lambda,\sigma)\right] & \text{if } \omega\le\sigma, \\ \mathcal{L}\left[i(\lambda-\sigma,\sigma)\right]\cdot\mathcal{L}\left[k(\lambda+\sigma,\omega)\right] - \mathcal{L}\left[j(\lambda,\sigma)\right]\cdot\mathcal{L}\left[k(\lambda,\omega)\right] & \text{if } \omega\ge\sigma. \end{cases}$$

Let $g(\lambda,\sigma,\omega) = \mathcal{L}^{-1}\left[\overline{G}(\sigma,\omega,s)\right]$. Notice that the function $\overline{G}$ is defined only for $\sigma,\omega,s>0$. Note that function $K_\beta(\sigma s)$ is not defined for $\sigma=0$. Then $g(\lambda,\sigma,\omega)$ is defined for $\sigma,\omega>0$. Let the convolution be denoted as $*$. Then by convolution theorem for the Laplace transform (*Schiff*, 1999)

$$g(\lambda,\sigma,\omega) = \begin{cases} i(\lambda-\omega,\omega) * k(\lambda+\omega,\sigma) - j(\lambda,\omega) * k(\lambda,\sigma) & \text{if } \omega \le \sigma, \\ i(\lambda-\sigma,\sigma) * k(\lambda+\sigma,\omega) - j(\lambda,\sigma) * k(\lambda,\omega) & \text{if } \omega \ge \sigma. \end{cases}$$

Let us introduce the functions $f_1(\lambda,\sigma,\omega) = i(\lambda-\omega,\omega) * k(\lambda+\omega,\sigma)$, $f_2(\lambda,\sigma,\omega) = j(\lambda,\omega) * k(\lambda,\sigma)$ $f_3(\lambda,\sigma,\omega) = i(\lambda-\sigma,\sigma) * k(\lambda+\sigma,\omega)$ and $f_4(\lambda,\sigma,\omega) = j(\lambda,\sigma) * k(\lambda,\omega)$. Then in terms of these functions

$$g(\lambda,\sigma,\omega) = \begin{cases} f_1(\lambda,\sigma,\omega) - f_2(\lambda,\sigma,\omega) \text{ if } \omega \le \sigma, \\ f_3(\lambda,\sigma,\omega) - f_4(\lambda,\sigma,\omega) \text{ if } \omega \ge \sigma. \end{cases} \tag{3.19}$$

Let us consider the function f_1. As a convolution of preimages under Laplace transform,

$$f_1(\lambda,\sigma,\omega) = \int_0^\lambda i(u-\omega,\omega)k(\lambda-u+\omega,\sigma)du.$$

Substituting $v = u-\omega$, and then denoting the variable v as u we obtain

$$f_1(\lambda,\sigma,\omega) = \int_{-\omega}^{\lambda-\omega} i(u,\omega)k(\lambda-u,\sigma)du.$$

Let us determine the limits of this integral. Suppose $f_1(\lambda,\sigma,\omega) \neq 0$ for some (λ,σ,ω). Then for some choice of u, $i(u,\omega) \neq 0$ and $k(\lambda-u,\sigma) \neq 0$. Let us consider the unit step functions in the formulas for the functions i and k. Since, $i(u,\omega) \neq 0$, $\omega - |u| > 0$ and since $\omega > 0$ then $-\omega < u < \omega$. Since $k(\lambda-u,\sigma) \neq 0$, $\lambda - u > \sigma$ and then $u < \lambda-\sigma$. Also, notice that u is already restricted so that $u > -\omega$ and $u < \lambda-\omega$. Together the inequalities $u < \omega$ and $u < \lambda-\sigma$ give us that $u < \min\{\omega, \lambda-\sigma, \lambda-\omega\}$. Since $\omega \le \sigma$, $\lambda-\sigma \le \lambda-\omega$ and then $u < \min\{\omega, \lambda-\sigma\}$. Also the inequalities $u < \lambda-\sigma$ and $u > -\omega$ give us that $\lambda-\sigma > -\omega$. Then,

$$f_1(\lambda,\sigma,\omega) = \int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i(u,\omega)k(\lambda-u,\sigma)du \cdot \mathcal{U}(\lambda-\sigma+\omega). \tag{3.20}$$

Let us consider the function f_3. Notice that f_3 differs from f_1 just by interchanging of the variables σ and ω. Then interchanging the arguments σ and ω in the derived formula for f_1 given in equation (3.20) we have that

$$f_3 = \int_{-\sigma}^{\min\{\lambda-\omega,\sigma\}} i(u,\sigma)k(\lambda-u,\omega)du \cdot \mathcal{U}(\lambda-\omega+\sigma). \tag{3.21}$$

Next, let us consider the function f_2. As a convolution,

$$f_2 = \int_0^\lambda j(u,\omega)k(\lambda-u,\sigma)du.$$

Now we will determine the integration boundaries in the formula for f_2. Suppose $f_2(\lambda,\sigma,\omega) \neq 0$ for some triple of the arguments. Then for some u, $j(u,\omega) \neq 0$ and $k(\lambda-u,\sigma) \neq 0$. Let us pay attention to the unit step functions in the formulas for j and k. First, since $j(u,\omega) \neq 0$ the unit step function in the formula for j (3.13) gives us that $u > \omega$. Second, since $k(\lambda - u, \sigma) \neq 0$, the unit step function in the formula for k (3.17) gives us that $\lambda - u > \sigma$ and so $u < \lambda - \sigma$. Since u is restricted by 0 from below, and λ from above, $u > \max\{0, -\omega\}$ and $u < \min\{\lambda, \lambda - \sigma\}$. Since $\sigma, \omega > 0$ $u > -\omega$ and $u < \lambda - \sigma$. Together all these inequalities yield the formula

$$f_2 = \int_{\omega}^{\lambda-\sigma} j(u,\omega)k(\lambda - u, \sigma)du \cdot \mathcal{U}(\lambda - \sigma - \omega). \tag{3.22}$$

Let us consider the function f_4. Notice that f_4 differs from f_2 just by interchanging of the variables σ and ω. Then interchanging the arguments σ and ω in the derived formula for f_2, given in equation (3.22) we have that

$$f_4(\lambda, \sigma, \omega) = \int_{\sigma}^{\lambda-\omega} j(u,\sigma)k(\lambda - u, \omega)du \cdot \mathcal{U}(\lambda - \omega - \sigma). \tag{3.23}$$

Defining $g(\lambda, \sigma, \omega)$ as 0 for negative values of λ we have that for all positive values of λ and σ the solution of the linearized shallow water equation (2.19) has the form

$$\Phi(\lambda, \sigma) = \sigma^{-\beta} \left[\int_0^{\infty} \omega^{\beta+1} R(\omega) g(\lambda, \sigma, \omega) d\omega \right].$$

where for positive λ, σ and ω the formula for the function $g(\lambda, \sigma, \omega)$ is defined in (3.19).

3.4 Obtaining the formula for the solution of the linearized shallow water equation

Now we want to simplify the formula for Φ. In particular, we want to determine boundaries for integration along ω. Expressing the function g through the functions f_1, f_2, f_3 and f_4, and using equation (3.19), we have that for $\lambda, \sigma > 0$

$$\Phi = \sigma^{-\beta} \left[\int_0^{\sigma} \omega^{\beta+1} R(\omega) \left(f_1 - f_2\right) d\omega + \int_{\sigma}^{\infty} \omega^{\beta+1} R(\omega) \left(f_3 - f_4\right) d\omega \right].$$

Let us introduce the new functions h_1, h_2, h_3 and h_4 so that

$$\begin{array}{ll} h_1(\lambda, \sigma) = \sigma^{-\beta} \int_0^{\sigma} \omega^{\beta+1} R(\omega) f_1 d\omega, & h_2(\lambda, \sigma) = \sigma^{-\beta} \int_0^{\sigma} \omega^{\beta+1} R(\omega) f_2 d\omega, \\ h_3(\lambda, \sigma) = \sigma^{-\beta} \int_{\sigma}^{\infty} \omega^{\beta+1} R(\omega) f_3 d\omega, & h_4(\lambda, \sigma) = \sigma^{-\beta} \int_{\sigma}^{\infty} \omega^{\beta+1} R(\omega) f_4 d\omega. \end{array} \tag{3.24}$$

Let us investigate integration boundaries for each of these functions separately.

First, let us consider the function h_1. Let us state the following claim for this function.

Proposition 1. *The function $h_1(\lambda, \sigma)$ can be expressed through the formula*

$$h_1(\lambda, \sigma) = \sigma^{-\beta} \int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) \left[\int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i(u,\omega)k(\lambda - u, \sigma)du \right] d\omega. \tag{3.25}$$

Proof. Suppose, $h_1(\lambda, \sigma) \neq 0$ for some positive λ and σ. Then $f_1 \neq 0$ for some choice of ω. Let us estimate the range of ω. Since $f_1 \neq 0$ and by the formula (3.20), the unit step function $\mathcal{U}(\lambda - \sigma + \omega)$ does not equal 0. This implies that $\lambda - \sigma + \omega \geq 0$ and hence, $\omega > \sigma - \lambda$. Since ω is between 0 and σ, $\omega > 0$, so $\omega > \max\{\sigma - \lambda, 0\}$. On the other hand, ω is restricted by σ. Since $\lambda, \sigma > 0$, $\max\{\sigma - \lambda, 0\} < \sigma$. Therefore, ω is bounded by $\max\{\sigma - \lambda, 0\}$ from below and by σ from above. Thus

$$h_1(\lambda,\sigma) = \sigma^{-\beta} \int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) f_1 d\omega.$$

Substituting the function f_1 given (3.20), we have that for positive values of λ and σ

$$h_1(\lambda,\sigma) = \sigma^{-\beta} \int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) \int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i(u,\omega) k(\lambda-u,\sigma) \mathcal{U}(\lambda-\sigma+\omega) du d\omega.$$

Since we derived the restrictions for ω from the unit step function $\mathcal{U}(\lambda-\sigma+\omega)$ used in the formula for f_1 and applied these restriction for the integration with respect to the variable ω, we can eliminate this unit step function from this formula. This gives us

$$h_1(\lambda,\sigma) = \sigma^{-\beta} \int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) \left[\int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i(u,\omega) k(\lambda-u,\sigma) du \right] d\omega \text{ -}$$

which is the same result as formula (3.25). $\square$

A similar analysis of h_2, h_3 and h_4 gives us that for all $\lambda > 0$ and $\sigma > 0$,

$$h_2(\lambda,\sigma) = \sigma^{-\beta} \int_{0}^{\min\{\sigma,\lambda-\sigma\}} \omega^{\beta+1} R(\omega) \left[\int_{\omega}^{\lambda-\sigma} j(u,\ \omega) \cdot k(\lambda-u,\ \sigma) du \right] d\omega \cdot \mathcal{U}(\lambda-\sigma), \tag{3.26}$$

$$h_3(\lambda,\sigma) = \sigma^{-\beta} \int_{\sigma}^{\lambda+\sigma} \omega^{\beta+1} R(\omega) \left[\int_{-\sigma}^{\min\{\sigma,\lambda-\omega\}} i(u,\ \sigma) \cdot k(\lambda-u,\ \omega) du \right] d\omega, \tag{3.27}$$

$$h_4(\lambda,\sigma) = \sigma^{-\beta} \int_{\sigma}^{\lambda-\sigma} \omega^{\beta+1} R(\omega) \left[\int_{\sigma}^{\lambda-\omega} j(u,\ \sigma) \cdot k(\lambda-u,\ \omega) du \right] d\omega \cdot \mathcal{U}(\lambda-2\sigma). \tag{3.28}$$

Notice that these formulas give us the value of Φ only for positive values of λ and σ.

Let us consider Φ for $\lambda = 0$. The upper limits of all four functions h_{1-4} with respect to λ at $\lambda = 0$ is 0. Then, assuming that all the functions h_{1-4} are continuous with respect to λ for $\lambda > 0$, we have that for $\lambda = 0$ they all 0. Therefore, $\Phi(\lambda,\sigma)$ is 0 for $\lambda = 0$ (as it should be, in order to satisfy the initial conditions (2.26)). Let us consider Φ for $\sigma = 0$. Assuming that Φ is continuous on a positive σ-semiaxis, we can define Φ at $\sigma = 0$ as upper-limit of Φ at the points satisfying $\sigma = 0$. So,

$$\Phi(\lambda,\sigma)|_{\sigma=0} = \lim_{\sigma\to 0+0} \Phi(\lambda,\sigma).$$

Clearly, the defined function Φ for $\sigma = 0$ reflects the behavior of the non-breaking wave at the edge of flooding area. Next, since the solution has no physical sense for negative values of λ and σ, let Φ be defined for these values to be 0. Combining all the previous results , we have that

$$\Phi(\lambda,\sigma) = \begin{cases} h_1(\lambda,\sigma) - h_2(\lambda,\sigma) + h_3(\lambda,\sigma) - h_4(\lambda,\sigma) & \text{if } \sigma > 0 \text{ and } \lambda > 0, \\ \lim_{\sigma\to 0+0} \Phi(\lambda,\sigma) & \text{if } \sigma = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.29}$$

where h_{1-4} are given in equations (3.25)-(3.28) respectively, and functions used in these formulas are defined as:

$$R(\sigma) = \Phi_\lambda(\lambda,\sigma)|_{\lambda=0} = 2g\frac{m+1}{m}\eta|_{\lambda=0}(\lambda,\sigma)$$

(by equations (2.26) and (3.6)), and for all positive $\tau > 0$ (which equals either σ or ω),

$$i(\mu,\tau) = \frac{1}{\pi}\cdot\frac{\cos(\beta\arccos(-\mu/\tau))}{\sqrt{\tau^2-\mu^2}}\cdot\mathcal{U}(\tau-|\mu|),$$

$$j(\mu,\tau) = \frac{\sin(\pi\beta)}{\pi}\cdot\frac{\exp(-\beta\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}\mathcal{U}(\mu-\tau),$$

$$k(\mu,\tau) = \frac{\cosh(\beta\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}\mathcal{U}(\mu-\tau),$$

$\beta = 1/m$, $m \in (0,+\infty]$ is the power of the bay cross-section.

3.5 The formula for Φ with a different order of integration

Formula (3.29) for Φ has several disadvantages. To see at least some of them, let us recall the formula for (3.25) h_1. It is easy to check that the convolution in this formula (this convolution earlier denoted as f_1) blows up if ω equals $\lambda-\sigma$. Using the formulas for i and k given in (3.12), (3.17) and the formula for f_1 given in equation (3.20), we have that

$$f_1(\lambda,\sigma,\omega) = \frac{1}{\pi}\int_{-\omega}^{\min\{\lambda-\sigma,\omega\}}\frac{\cos(\frac{1}{m}\cdot\arccos(-u/\omega))\cdot\cosh(\frac{1}{m}\cdot\cosh^{-1}((\lambda-u)/\sigma))}{\sqrt{\omega^2-u^2}\sqrt{(\lambda-u)^2-\sigma^2}}du\cdot\mathcal{U}(\lambda-\sigma+\omega).$$

Clearly, both roots in the denominator converge to 0 if $\lambda-\sigma=\omega$ and u converges to ω. This means that, the improper integral can diverge if $\lambda-\sigma=\omega$. For example, if $m\to+\infty$, (which corresponds to the plane beach), this integral diverges. This gives us doubts about the convergence of the whole repeated integral which is in the formula for the function $h_1(\lambda,\sigma)$, (see equation (3.25)). Fortunately, the asymptotic analysis of this improper repated integral shows that this integral converges. However, the divergence of the improper integral in the formula for f_1 makes the numerical computation of the repeated integral for the function $h_1(\lambda,\sigma)$ especially difficult.

A similar problem exists for the function $h_3(\lambda,\sigma)$ given in equation (3.27).

On the other hand, if we compute the integral of a function which involves only one of functions within i and k but not both, the other factors of this function are finite, then for any finite area of integration this integral will converge! Clearly, for any positive σ, and any finite a and b, the improper integrals $\int_a^b i(u,\sigma)du$ $\int_a^b j(u,\sigma)du$ is convergent! (It easy to check by making the trigonometric substitution for u in the first case, and hyperbolic substitution in the second case).

How to solve these problems?

We would like to "separate" unbounded functions i and k which exists in the formulas for h_1 and h_3. In order to do this we would like to change the orders of the integrations for the repeated integrals in the formulas for both functions. The other reasons why we would like to change the order of integration can be for the computation of some limits of Φ, including partial derivatives and limits for σ converging to 0. Or maybe we would just like to have some more freedom in representing the function Φ! Then we would also want to change the orders of integration in the functions h_2 and h_4 (given in (3.26) and (3.28)).

Can we change the order of integrations without violation of the equality? As one can see from the formulas for the functions (3.25)-(3.28), all domains of integration in these formulas are finite. For these cases, the Tonelli-Hobson Theorem (see (*Ray*, 1988) statement 7.5.13) answers this question. Part of the theorem is as follows:

Proposition 2. *If A and B are complete spaces of the finite measure, and if for a function $f(x, y)$ both repeated integrals $\int_A \left(\int_B |f(x, y)|\right) dx$ and $\int_B \left(\int_A |f(x, y)|\right) dy$ are finite, then in particular $\int_A \left(\int_B f(x, y)\right) dx = \int_B \left(\int_A f(x, y)\right) dy$.*

As one can check, this statement tells us that we can change the orders of integration in the formulas for h_1, h_2, h_3 and h_4 without change of value. So, now we are going to do it!

Let us consider the function h_1. But we have that

$$h_1(\lambda, \sigma) = \int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) f_1(\lambda, \sigma, \omega) d\omega,$$

where $f_1(\lambda, \sigma, \omega)$ is the convolution of the function $i(\lambda, \omega)$ with the function $k(\lambda, \sigma)$ with respect to the argument λ:

$$f_1(\lambda, \sigma, \omega) = i(\lambda, \omega) * k(\lambda, \sigma).$$

Then

$$h_1(\lambda, \sigma) = \sigma^{-\beta} \int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) \left[\int_{-\infty}^{\infty} i(u, \omega) \cdot k(\lambda - u, \sigma) \cdot \mathcal{U}(\lambda - \sigma) du\right] d\omega.$$

Let $h_1'(\lambda, \sigma)$ be the function obtained from h_1 by exchanging the orders of integration. Then

$$h_1'(\lambda, \sigma) = \sigma^{-\beta} \int_{-\infty}^{\infty} \left[\int_{\max\{\sigma-\lambda,0\}}^{\sigma} \omega^{\beta+1} R(\omega) i(u, \omega) d\omega\right] k(\lambda - u, \sigma) du.$$

The analysis of integral boundaries similar to the one on the integral boundaries of h_1 in the proposition 1 yields the result

$$h_1'(\lambda, \sigma) = \sigma^{-\beta} \int_{-\sigma}^{\min\{\sigma,\lambda-\sigma\}} \left[\int_{\min\{\sigma-\lambda,|u|\}}^{\sigma} \omega^{\beta+1} R(\omega) i(u, \omega) d\omega\right] k(\lambda - u, \sigma) du. \tag{3.30}$$

Let $h_2'(\lambda, \sigma)$, $h_3'(\lambda, \sigma)$ and $h_4'(\lambda, \sigma)$ be the functions derived from the functions $h_2(\lambda, \sigma)$ $h_3(\lambda, \sigma)$ and $h_4(\lambda, \sigma)$ respectively by exchanging the orders of integrations. A similar analysis of the formulas for the functions $h_2(\lambda, \sigma)$ $h_3(\lambda, \sigma)$ and $h_4(\lambda, \sigma)$ (which are given in equations (3.25), (3.27), (3.28)) yields the result

$$h_2'(\lambda,\sigma) = \sigma^{-\beta}\int_0^{\lambda-\sigma}\left[\int_0^{\min\{\sigma,u\}}\omega^{\beta+1}R(\omega)j(u,\omega)d\omega\right]k(\lambda-u,\sigma)du\cdot\mathcal{U}(\lambda-\sigma), \tag{3.31}$$

$$h_3'(\lambda,\sigma) = \sigma^{-\beta}\int_{-\sigma}^{\min\{\sigma,\lambda-\sigma\}}\left[\int_\sigma^{\lambda-u}\omega^{\beta+1}R(\omega)k(\lambda-u,\omega)d\omega\right]i(u,\sigma)du, \tag{3.32}$$

$$h_4'(\lambda,\sigma) = \sigma^{-\beta}\int_\sigma^{\lambda-\sigma}\left[\int_\sigma^{\lambda-u}\omega^{\beta+1}R(\omega)k(\lambda-u,\omega)d\omega\right]j(u,\sigma)du\cdot\mathcal{U}(\lambda-2\sigma). \tag{3.33}$$

Hence, $\Phi(\lambda,\sigma)$, the solution of equation (2.19) can be represented as

$$\Phi(\lambda,\sigma) = \begin{cases} h_1'(\lambda,\sigma) - h_2'(\lambda,\sigma) + h_3'(\lambda,\sigma) - h_4'(\lambda,\sigma) & \text{if } \lambda,\sigma > 0, \\ \lim_{\sigma\to 0+0}\Phi(\lambda,\sigma) & \text{if } \sigma = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.34}$$

where $h_1'(\lambda,\sigma)$, $h_2'(\lambda,\sigma)$, $h_3'(\lambda,\sigma)$ and $h_4'(\lambda,\sigma)$ are given in the formulas (3.30)-(3.33) respectively.

Chapter 4
Verification of the solution of the linearized equation, obtained by the Laplace transform, for particular cases

4.1 Case 1. Plane beach case

Let us recall the shallow water problem for the plane beach. Originally this problem was considered and solved by Carrier and Greenspan (*Carrier and Greenspan*, 1958). They used the hodograph type transformation, and converted the shallow water equation to the partial differential equation similar to equation (2.19). In the variables we are using, their equation is

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{1}{\sigma}\Phi_{\sigma} = 0. \tag{4.1}$$

The way in which they solved this equation was the use of the Fourier-Bessel transform (*Farlow*, 1993) with respect to the variable σ.

Recall, that a plane beach bay is a bay which has the constant slope in the offshore direction and a flat cross-section of finite width bounded by vertical walls (see Figure 1.4 (a)). As one can see, it is a degenerate case of a sloping inclined bay with a cross-section parameterized by the power function $z = |cy|^m$, where m converges to $+\infty$. Let us recall the linearized shallow water differential equation which, in terms of the potential Φ, describes the shallow water problem for these bays (equation (2.19)). This equation is

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{m+2}{m\sigma}\Phi_{\sigma} = 0.$$

Converging m to infinity gives us equation (4.1). This confirms the fact that the shallow water problem in the plane beach bay is a special case of the shallow water problem in the inclined bay, with a cross-section parameterized by the power function $z = |cy|^m$, $c > 0$. (We were usually writing the function $z = c|y|^m$. But as one can see, for a finite m the representations $z = |cy|^m$ and $z = c|y|^m$ are equivalent.)

Now we would like to show, that the solution of equation (4.1) which has been derived through the Laplace transformation with respect to the variable λ is the same as the solution that can be derived through the Fourier-Bessel transformation with respect to the variable σ. Recall that the solution for equation (2.19) (and consequently for equation (4.1) obtained through Laplace transformation) has two different orders of integration (see the formulas (3.29) and (3.34)). Let us for simplicity consider only one order of integration, the solution (3.29), and compare it with the solution which we will derive through the Fourier-Bessel transform.

4.1.1 Method 1. Solving by Laplace transform.

Let us consider the function $\Phi(\lambda, \sigma)$ given in equation (3.29) for $m = +\infty$. Notice that it solves equation (4.1), the linearized shallow water equation for the plane beach.

Recall, that this function is the linear combination of the functions h_{1-4} (see equations (3.25)-(3.28)), that depend on the functions i, j, and k (given in equations (3.12)-(3.17)). At the same time i, j and k functions depend on the parameter β, where β is defined as $\frac{1}{m}$. For the plane beach β equals to zero.

As one can see, when $\beta = 0$, $j \equiv 0$ and consequently the functions h_2 and h_4 also equal to zero. Let us express the functions i and k for $\beta = 0$. Since $\cos(0) = 1$ and $\cosh(0) = 1$, for any $\tau > 0$,

$$i|_{\beta=0}(\mu,\tau) = \frac{1}{\pi}\frac{\mathcal{U}(\tau - |\mu|)}{\sqrt{\tau^2-\mu^2}}, \text{ and } k|_{\beta=0}(\mu,\tau) = \frac{\mathcal{U}(\mu-\tau)}{\sqrt{\mu^2-\tau^2}}. \tag{4.2}$$

Combining these results we have that Φ, obtained by the spectral method has the form

$$\Phi(\lambda,\sigma) = \begin{cases} h_1|_{\beta=0}(\lambda,\sigma) + h_3|_{\beta=0}(\lambda,\sigma) & \text{if } \sigma > 0\ \lambda \geq 0, \\ \lim_{\sigma\to 0+0}\Phi(\lambda,\sigma) & \text{if } \sigma = 0,\ \lambda \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

where

$$h_1|_{\beta=0}(\lambda,\sigma) = \int_{\max\{0,\sigma-\lambda\}}^{\sigma} \omega R(\omega)\left[\int_{-\omega}^{\min\{\omega,\lambda-\sigma\}} i|_{\beta=0}(u,\omega)\cdot k|_{\beta=0}(\lambda-u,\sigma)du\right]d\omega,$$

$$h_3|_{\beta=0}(\lambda,\sigma) = \int_{\sigma}^{\lambda+\sigma} \omega R(\omega)\left[\int_{-\sigma}^{\min\{\sigma,\lambda-\omega\}} i_{\beta=0}(u,\sigma)|\cdot k_{\beta=0}(\lambda-u,\omega)|du\right]d\omega,$$

and the functions $i|_{\beta=0}(\mu,\tau)|\ k|_{\beta=0}(u,\tau)$ are given by the system (4.2).

4.1.2 Method 2. Solving by Fourier-Bessel transform after Carrier-Greenspan.

Now, we will solve equation (4.1) satisfying boundary (2.27) and initial (2.26) conditions by using the Fourier-Bessel transform with respect to σ. We will actually reproduce the work, done by Carrier and Greenspan. Since we define Φ only for nonnegative values of σ, then this this kind of transformation can be applied to equation (4.1).

Let

$$\overline{\Phi}(\lambda,\rho) = \int_0^\infty \sigma J_0(\rho\sigma)\Phi(\lambda,\sigma)d\sigma$$

be the image of Φ under the Fourier-Bessel transform, where $J_0(\rho\sigma)$ is the Bessel function of the first kind. Assuming that Φ behaves such that we can interchange the transformation with differentiation without change the result, we have that

$$\overline{\Phi}_{\lambda\lambda}(\lambda,\rho) = \int_0^\infty \sigma J_0(\rho\sigma)\Phi_{\lambda\lambda}(\lambda,\sigma)d\sigma.$$

Let us apply the Fourier-Bessel transformation to equation (4.1). In order to do it, let us first simplify this equation. Multiplication of this equation by sigma gives

$$\sigma\Phi_{\lambda\lambda} - (\sigma\Phi_\sigma)_\sigma = 0. \tag{4.4}$$

Further multiplying of this equation (4.4) by $J_0(\rho\sigma)$ and integrating in terms of variableσ from 0 to ∞ yields

$$\int_0^\infty \sigma J_0(\rho\sigma)\Phi_{\lambda\lambda} d\sigma - \int_0^\infty J_0(\rho\sigma)(\sigma\Phi_\sigma)_\sigma d\sigma,$$

where the first integral is equal to $\overline{\Phi}_{\lambda\lambda}$. Replacing the second integral by the right side we have that

$$\overline{\Phi}_{\lambda\lambda} = \int_0^\infty \sigma J_0(\rho\sigma)(\sigma\Phi_\sigma)_\sigma d\sigma.$$

The integration by parts gives

$$\overline{\Phi}_{\lambda\lambda} = (J_0(\rho\sigma)\sigma\Phi_\sigma)\big|_0^\infty - \int_0^\infty (J_0(\rho\sigma))_\sigma \sigma\Phi_\sigma d\sigma.$$

Assuming, that Φ_σ decays sufficiently fast so that $\sigma\Phi_\sigma \to 0$ and since the Bessel function is bounded on its domain (*Abramowitz and Stegun*, 1965), we have that $((J_0)(\rho\sigma)\sigma\Phi_\sigma)_{\sigma\to\infty} \to 0$. Since also $\Phi_\sigma \to 0$ for $\sigma \to 0$, we have that

$$\overline{\Phi}_{\lambda\lambda} = -\int_0^\infty (J_0(\rho\sigma))_\sigma \sigma\Phi_\sigma d\sigma.$$

Why should the product $\sigma\Phi_\sigma$ converge to 0 for σ converging $+\infty$? Recall from boundary conditions (2.27) that $\Phi(\lambda,\sigma)$ converge to 0 for $\sigma \to \infty$. Then as one can notice, if the absolute value of the function $\Phi(\lambda,\sigma)$ has an asymptotic $A\sigma^{-\gamma}$ for some $A, \gamma > 0$, then the absolute value of the function $\sigma\Phi_\sigma(\lambda,\sigma)$ has the asymptotic $A \cdot \gamma\sigma^{-\gamma}$. So, in this case the function $|\sigma\Phi_\sigma(\lambda,\sigma)|$ has the asymptotic of the same power as the function $|\Phi(\lambda,\sigma)|$ and so $\sigma\Phi_\sigma(\lambda,\sigma)$ should also converge to 0. It shows us that the assumption of that $\sigma\Phi_\sigma \to 0$ for $\sigma \to \infty$ is very reliable.

Now let us further simplify the integral. Integrating by parts again gives us that

$$\overline{\Phi}_{\lambda\lambda} = -\sigma(J_0(\rho\sigma))_\sigma \Phi\big|_0^\infty + \int_0^\infty (\sigma(J_0(\rho\sigma))_\sigma)_\sigma \Phi d\sigma,$$

it follows from the boundary conditions (2.27) that the first term is equal to 0 and we now have

$$\overline{\Phi}_{\lambda\lambda} = \int_0^\infty \Phi(\sigma(J_0(\rho\sigma))_\sigma)_\sigma d\sigma.$$

Let us consider $(\sigma(J_0(\rho\sigma))_\sigma)_\sigma$ separately. Let us denote it as $Z(\rho,\sigma)$. Then

$$\overline{\Phi}_{\lambda\lambda} = \int_0^\infty \Phi(\lambda,\sigma)Z(\rho,\sigma)d\sigma.$$

Let us simplify $Z(\rho,\sigma)$. By the product rule,

$$Z(\rho,\sigma) = (J_0(\rho\sigma))_\sigma + \sigma(J_0(\rho\sigma))_{\sigma\sigma}.$$

Substituting $x = \rho\sigma$ we have,

$$Z(\rho,x) = \rho(J_0'(x) + xJ_0''(x)).$$

Recall that $J_0(x)$ solves the Bessel's equation $x^2y'' + xy' + x^2y = 0$ (*Abramowitz and Stegun*, 1965). Plugging $J_0(x)$ into this equation gives us the identity:

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0,$$

and this identity implies

$$Z(\rho, x) = -\rho x J_0(x).$$

Returning to the variables ρ and σ we can write

$$Z(\rho, x) = -\rho^2 \sigma J_0(\rho\sigma).$$

Putting this equation back into integral we have, that

$$\overline{\Phi}_{\lambda\lambda} = -\rho^2 \int_0^\infty \sigma J_0(\rho\sigma)\Phi d\sigma,$$

where the integral on the right side is equal to the image of Φ under Fourier Bessel transform - $\overline{\Phi}(\lambda, \rho)$. Then

$$\overline{\Phi}_{\lambda\lambda} = -\rho^2\overline{\Phi}$$

and hence, we obtain equation

$$\overline{\Phi}_{\lambda\lambda} + \rho^2\overline{\Phi} = 0. \tag{4.5}$$

Next, applying the transformation to the initial conditions (2.26) modified by equation (4.1) give us the initial conditions for equation (4.5). They are

$$\begin{cases} \overline{\Phi}(\lambda, \rho)|_{\lambda=0} = 0, \\ \overline{\Phi}_\lambda(\lambda, \rho)|_{\lambda=0} = \int_0^\infty \omega R(\omega) J_0(\rho\omega) d\omega, \end{cases}$$

where by equation (3.6), $R(\omega) = 2g\eta(\lambda, \omega)|_{\lambda=0}$.

By solving the obtained initial value problem, can find the formula for $\overline{\Phi}$.As one can check,

$$\overline{\Phi}(\lambda, \rho) = \begin{cases} \int_0^\infty \omega R(\omega) J_0(\rho\omega) d\omega \frac{\sin(\rho\lambda)}{\rho} & \lambda > 0 \text{ and } \rho > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In order to find Φ we need to find the preimage of $\overline{\Phi}$ under the Fourier-Bessel transform. Let Φ be the preimage of $\overline{\Phi}$. Then for $\lambda > 0$ and $\sigma > 0$,

$$\Phi(\lambda, \sigma) = \int_0^\infty J_0(\rho\sigma) \sin(\rho\lambda) \int_0^\infty \omega J_0(\rho\omega) R(\omega) d\omega d\rho. \tag{4.6}$$

Next, we would like to change the order of integration in the formula for Φ for $\lambda, \sigma > 0$. Let us assume we can change the order of the integration without changing the result. We always can check the legitimacy of this assumption later. Accepting this assumption for now we have

$$\Phi(\lambda, \sigma) = \int_0^\infty \omega R(\omega) \left[\int_0^\infty J_0(\rho\sigma) \sin(\rho\lambda) J_0(\rho\omega) d\rho \right] d\omega.$$

Let us introduce $I(\sigma,\omega,\lambda)$ so for all $\lambda,\sigma,\omega>0$

$$I(\sigma,\omega,\lambda)=\int_0^\infty J_0(\rho\sigma)\sin(\rho\lambda)J_0(\rho\omega)d\rho. \tag{4.7}$$

Then

$$\Phi(\lambda,\sigma)=\int_0^\infty \omega R(\omega)I(\sigma,\omega,\lambda)d\omega.$$

Notice, that we can represent Φ as

$$\begin{aligned}\Phi(\lambda,\sigma) &= \Phi_1(\lambda,\sigma)+\Phi_2(\lambda,\sigma), &(4.8)\\ \text{where} \quad & \Phi_1(\lambda,\sigma)=\int_0^\sigma \omega R(\omega)I|_{\sigma\geq\omega}(\sigma,\omega,\lambda)d\omega \text{ and} &(4.9)\\ & \Phi_2(\lambda,\sigma)=\int_\sigma^\infty \omega R(\omega)I|_{\sigma\leq\omega}(\sigma,\omega,\lambda)d\omega. &(4.10)\end{aligned}$$

Next, we need to simplify the formulas for Φ_1 and Φ_2. In order to do this, first, let us consider the function $I(\sigma,\omega,\lambda)$. By the formula derived in (*Carrier et al.*, 2003) for all positive λ, σ and ω,

$$I|_{\sigma\geq\omega}(\sigma,\omega,\lambda)=\frac{1}{\pi}\int_{-1}^1\frac{\mathcal{U}(\lambda-\omega\xi-\sigma)}{\sqrt{1-\xi^2}\sqrt{(\lambda-\omega\xi)^2-\sigma^2}}d\xi.$$

Let us make the substitution $u=\omega\xi$. This substitution yields,

$$I|_{\sigma\geq\omega}(\sigma,\omega,\lambda)=\frac{1}{\pi}\int_{-\omega}^\omega\frac{\mathcal{U}(\lambda-u-\sigma)}{\sqrt{\omega^2-u^2}\sqrt{(\lambda-u)^2-\sigma^2}}du.$$

Let us modify this formula. Suppose, $I|_{\sigma\geq\omega}(\sigma,\omega,\lambda)\neq 0$. Then in this formula $-\omega<u<\omega$ which means that $|u|<\omega$. Consequently, $\mathcal{U}(\omega-|u|)\neq 0$. Since $\mathcal{U}(\lambda-u-\sigma)\neq 0$, $\lambda-u-\sigma>0$ or in other words u is restricted by $\lambda-\sigma$ from above: $u<\lambda-\sigma$. Together the inequalities $-\omega<u<\omega$ and $u<\lambda-\sigma$ give us $-\omega<u<\min\{\lambda-\sigma,\omega\}$ and $\lambda-\sigma>-\omega$. Applying these inequalities to the integral, we have

$$I|_{\sigma\geq\omega}(\sigma,\omega,\lambda)=\frac{1}{\pi}\int_{-\omega}^{\min\{\lambda-\sigma,\omega\}}\frac{\mathcal{U}(\omega-|u|)}{\sqrt{\omega^2-u^2}}\cdot\frac{\mathcal{U}(\lambda-u-\sigma)}{\sqrt{(\lambda-u)^2-\sigma^2}}du\cdot\mathcal{U}(\lambda-\sigma+\omega).$$

Recalling the functions i and k and the system (4.2), notice, that the factors $\frac{\mathcal{U}(\omega-|u|)}{\sqrt{\omega^2-u^2}}$ and $\frac{\mathcal{U}(\lambda-u-\sigma)}{\sqrt{(\lambda-u)^2-\sigma^2}}$ are equal to the functions $i|_{\beta=0}(u,\omega)$ and $k|_{\beta=0}(\lambda-u,\sigma)$ respectively. So, we can write this formula as

$$I|_{\sigma\geq\omega}(\sigma,\omega,\lambda)=\int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i|_{\beta=0}(u,\omega)\cdot k|_{\beta=0}(\lambda-u,\sigma)du\cdot\mathcal{U}(\lambda-\sigma+\omega).$$

Similarly, let us consider the function $I(\sigma,\omega,\lambda)$ for $\sigma\leq\omega$. First, analyzing function (4.7) notice, that we can interchange the arguments of this function σ and ω without changing the result. Interchanging these arguments we simplify the problem of evaluating $I|_{\sigma\leq\omega}(\sigma,\omega,\lambda)$ to the problem of evaluating the function $I|_{\omega\geq\sigma}(\omega,\sigma,\lambda)$ and we obtain the result

$$I|_{\omega\geq\sigma}(\omega,\sigma,\lambda)=\int_{-\sigma}^{\min\{\lambda-\omega,\sigma\}} i|_{\beta=0}(u,\sigma)\cdot k|_{\beta=0}(\lambda-u,\omega)du\cdot\mathcal{U}(\lambda+\sigma-\omega).$$

Substituting these results for the function I into the formulas for Φ_1 and Φ_2 given in equations (4.9)-(4.10) we obtain

$$\Phi_1(\lambda,\sigma) = \int_0^{\sigma} \omega R(\omega) \int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i|_{\beta=0}(u,\omega)\cdot k|_{\beta=0}(\lambda-u,\sigma)du \cdot \mathcal{U}(\lambda-\sigma+\omega)d\omega,$$

and

$$\Phi_2(\lambda,\sigma) = \int_{\sigma}^{\infty} \omega R(\omega) \int_{-\sigma}^{\min\{\lambda-\omega,\sigma\}} i|_{\beta=0}(u,\sigma)\cdot k|_{\beta=0}(\lambda-u,\omega)du \cdot \mathcal{U}(\lambda+\sigma-\omega)d\omega.$$

Notice, that the function $\mathcal{U}(\lambda-\sigma+\omega)$ in the formula for Φ_1 gives us $\omega > \sigma-\lambda$ and the function $\mathcal{U}(\lambda+\sigma-\omega)$ in the formula for Φ_2 gives us $\omega < \lambda+\sigma$. Since we consider both functions only for $\lambda > 0$, we can write these functions as

$$\Phi_1(\lambda,\sigma) = \int_{\max\{0,\sigma-\lambda\}}^{\sigma} \omega R(\omega) \left[\int_{-\omega}^{\min\{\lambda-\sigma,\omega\}} i|_{\beta=0}(u,\omega)\cdot k|_{\beta=0}(\lambda-u,\sigma)du\right] d\omega \tag{4.11}$$

and

$$\Phi_2(\lambda,\sigma) = \int_{\sigma}^{\lambda+\sigma} \omega R(\omega) \left[\int_{-\sigma}^{\min\{\lambda-\omega,\sigma\}} i|_{\beta=0}(u,\sigma)\cdot k|_{\beta=0}(\lambda-u,\omega)du\right] d\omega. \tag{4.12}$$

But, as one can see, these formulas also work for $\lambda = 0$, producing the value of 0. Recall that for positive λ,σ we defined Φ as sum of Φ_1 and Φ_2 (equation (4.8)). By defining Φ to be continuous from the right for $\sigma = 0$ and extended as 0 for $\lambda < 0$ and $\sigma < 0$ we have

$$\Phi(\lambda,\sigma) = \begin{cases} \Phi_1(\lambda,\sigma)+\Phi_2(\lambda,\sigma) & \text{if } \lambda \geq 0 \text{ and } \sigma > 0, \\ \lim_{\sigma\to 0+0}\Phi(\lambda,\sigma) & \text{if } \sigma = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the formulas for $\Phi_1(\lambda,\sigma)$ and $\Phi_2(\lambda,\sigma)$ are given in equations (4.11) (4.12). Clearly, this is the same result as the result obtained through Laplace transformation given in equation (4.3).

Recall that we changed the order of integration when we computed the inverse Fourier-Bessel transform (in equation (4.6)). After making this change we obtained the result given in equation (4.3). However, the result (4.3) was also obtained by a different way. Then most likely the interchange of the orders of integration in the formula (4.6) did not affect on the result. Also most likely interchange of the orders of transformation and differentiation by did not affect the result too. Otherwise, that formula obtained by two different ways would be almost unlikely to coincide.

Therefore, Φ, the solution of equation (4.1), obtained by the Fourier-Bessel transform is the same as the solution obtained by the Laplace transform, given in (4.3). Hence, the solution of the linearized shallow water problem obtained by Laplace transform and represented in formula (3.29) is most likely correct for the plane beach case.

4.2 Case 2. An inclined bay with the parabolic cross-section

Let us recall the shallow water problem for an inclined bay with the cross-section parameterized by $z = c|y|^2$.

Earlier, this problem was considered by Didenkulova and Pelinovsky in (*Didenkulova and Pelinovsky*, 2011). In their work they used the hodograph type transformation (*Zahibo et al.*, 2006) to derive the corresponding linear partial differential equation in terms of Φ. Notice, that it is a particular case of the shallow water problem with the sloping bay and cross-section parameterized by $z = c|y|^m$, where $m = 2$. Let us recall the equation (2.19). Substituting $m = 2$ we obtain equation

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{2}{\sigma}\Phi_\sigma = 0. \tag{4.13}$$

This equation has the same boundary and initial conditions as (2.26) and (2.27). It was solved by Didenkulova and Pelinovsky for nonnegative σ and λ by the D'Alembert method (*Farlow*, 1993). We want to compare this solution with the solution obtained by the Laplace transform.

4.2.1 Method 1. Solving by Laplace transform

First, we will write the solution of equation (4.13) obtained by the Laplace transform. As one can see, equation (4.13) is a particular case of equation (2.19) for $m = 2$. Then the solution of equation (4.13) is a particular case of the solution of equation (2.19) for $m = 2$.

Let us consider the solution of equation (2.19). Recall that we obtained it with two orders of integration. We are going choose it as it is given in equation (3.29). Then let us pick this formula and consider it for $m = 2$.

Recall, that it is a linear combination of the functions h_{1-4} given in equations (3.25)-(3.28):

$$\Phi(\lambda, \sigma) = h_1(\lambda, \sigma) - h_2(\lambda, \sigma) + h_3(\lambda, \sigma) - h_4(\lambda, \sigma).$$

Let us recall the functions f_{1-4} given in equations (3.20)-(3.23) and notice, that the functions h_{1-4} can be expressed through functions f_{1-4} and the functions f_{1-4} depend on functions i, j, k. We will simplify the solution of equation (4.13) in the following way:

(1) We simplify i, j, k for $m = 2$

(2) Using simplified i, j, k, we simplify f_{1-4}.

(3) Through the simplified functions f_{1-4} we simplify the functions h_{1-4}.

Let us also recall functions i, j and k given in equations (3.12) (3.13) and (3.17) respectively. Notice, that they depend on parameter β defined as $\frac{1}{m}$. Since we consider $m = 2$, then $\beta = \frac{1}{2}$. As one can easy check, for $\beta = 1/2$, and $\tau > 0$

$$i|_{\beta=\frac{1}{2}}(\mu, \tau) = \frac{1}{\pi} \frac{\cos(\frac{1}{2}\arccos(-\mu/\tau))}{\sqrt{\tau^2 - \mu^2}} \cdot \mathcal{U}(\tau - |\mu|),$$

$$j|_{\beta=\frac{1}{2}}(\mu,\tau) = \frac{1}{\pi}\frac{\exp(-\frac{1}{2}\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}\cdot\mathcal{U}(\mu-\tau), \text{ and}$$

$$k|_{\beta=\frac{1}{2}}(\mu,\tau) = \frac{\cosh(\frac{1}{2}\cosh^{-1}(\mu/\tau))}{\sqrt{\mu^2-\tau^2}}\cdot\mathcal{U}(\mu-\tau).$$

Now we need to simplify these formulas. During this simplification we will eliminate hyperbolic, trigonometric functions and their inverses from the formula.

Consider the function $i|_{\beta=\frac{1}{2}}(\mu,\tau)$. Let us substitute $\phi = \frac{1}{2}\arccos(-\mu/\tau))$ in the numerator and consider the $\cos(\phi)$. We would like to represent it in the terms of $\cos(\arccos(-\mu/\tau))$. In this case we will be able to annihilate the cosine and its inverse. By the half argument formula for the cosine,

$$\cos(\phi) = \pm\sqrt{\frac{1+\cos(2\phi)}{2}}.$$

Since the range of arccosine is $[0,\pi]$, the range of ϕ is $[0,\pi/2]$. Recall that cosine is nonnegative on this interval. Then in all possible cases,

$$\cos(\phi) = \sqrt{\frac{1+\cos(2\phi)}{2}}.$$

Substituting ϕ back, we have that

$$\cos\left(\frac{1}{2}\arccos(-\mu/\tau)\right) = \sqrt{\frac{1+\cos(\arccos(-\mu/\tau))}{2}}.$$

Eliminating the cosine and its inverse from the formula, and simplifying we have that

$$\cos\left(\frac{1}{2}\arccos(-\mu/\tau)\right) = \frac{\sqrt{\tau-\mu}}{\sqrt{2\tau}}.$$

Substituting this expression back into the formula for $i(\tau,\mu)|_{\beta=\frac{1}{2}}$ gives us

$$i|_{\beta=\frac{1}{2}}(\mu,\tau) = \frac{1}{\pi}\frac{\sqrt{\tau-\mu}}{\sqrt{2\tau}\sqrt{\tau^2-\mu^2}}\cdot\mathcal{U}(\tau-|\mu|).$$

Dividing numerator and denominator by $\sqrt{\tau-u}$, we obtain

$$i|_{\beta=\frac{1}{2}}(\mu,\tau) = \frac{1}{\pi}\frac{\mathcal{U}(\tau-|\mu|)}{\sqrt{2\tau}\sqrt{\tau+\mu}}.$$

Next, let us consider the function $k|_{\beta=1/2}(\mu,\tau)$.

First, we need to explain the notation $\cosh^{-1}(x)$ in this formula. This notation implies the function which is inverse to the function $\cosh(x)$. For each real number in the ray $[1,\infty)$ the function $\cosh^{-1}(x)$ is defined so that for each $y\in[1,\infty)$, $\cosh(y)\geq 0$.

Let us return to the function $k|_{\beta=1/2}$ and consider its numerator: $\cosh(\frac{1}{2}\cosh^{-1}(\mu/\tau))$ with $\tau>0$ and $\mu>\tau$. Let us in particular investigate the expression $\cosh^{-1}(\mu/\tau)$. Notice, that since $\tau>0$ and $\mu>\tau$, $\mu/\tau>1$ and then $\cosh^{-1}(\mu/\tau)$ is real and positive. Let us come back to the expression

$\cosh(\frac{1}{2}\cosh^{-1}(\mu/\tau))$ with $\tau > 0$ and $\mu > \tau$. Similarly, as we did to the function $i|_{\beta=\frac{1}{2}}$, we would like annihilate the hyperbolic cosine and its inverse in this expression. Let $\phi = \frac{1}{2}\cosh^{-1}(\mu/\tau)$. Then the numerator is equal to $\cosh(\phi)$. By the formula for the half-argument for the hyperbolic cosine,

$$\cosh(\phi) = \sqrt{\frac{1+\cosh(2\phi)}{2}}.$$

Applying this identity we have, that

$$\cosh\left(\frac{1}{2}\cosh^{-1}(\mu/\tau)\right) = \sqrt{\frac{1+\cosh(\cosh^{-1}(\mu/\tau))}{2}},$$

where the functions $\cosh(x)$ $\cosh^{-1}(x)$ annihilate each other. Eliminating these functions from the formula and simplifying, we obtain

$$\cosh\left(\frac{1}{2}\cosh^{-1}(\mu/\tau)\right) = \sqrt{\frac{\tau+\mu}{2\tau}}.$$

Substituting this expression back into the formula for $k|_{\beta=1/2}$ and simplifying in the same way as we have done for the function $i|_{\beta=1/2}$, we obtain that

$$k|_{\beta=1/2} = \frac{\mathcal{U}(\mu-\tau)}{\sqrt{2\tau}\sqrt{\mu-\tau}}.$$

Let us consider the function $j|_{\beta=\frac{1}{2}}(\mu,\tau)$. We will consider it for $\tau > 0$ and $\mu > \tau$ (otherwise this function is trivial). Let us consider the expression in the numerator, $\exp(-\frac{1}{2}\cosh^{-1}(u/\tau))$. Let $\phi = \frac{1}{2}\cosh^{-1}(u/\tau)$. Then this expression turns into $\exp(-\phi)$. Clearly, by formulas of hyperbolic functions

$$\exp(-\phi) = \cosh(\phi) - \sinh(\phi).$$

Now, to be able to annihilate the function $\cosh^{-1}$ in the numerator in this case, first, we must express both functions $\cosh(\phi)$ and $\sinh(\phi)$ in suitable forms - in terms of the function $\cosh(2\phi)$. Notice that μ and τ and ϕ have the same restrictions on the ranges as they had for the function $k|_{\beta=\frac{1}{2}}(\mu,\tau)$. Then $\phi > 0$, and $\cosh(\phi)$ is expressed through $\cosh(2\phi)$ in same way as we computed it for the function $k|_{\beta=1/2}$. Let us express the function $\sinh(\phi)$ in terms of $\cosh(2\phi)$. By the formula for the half-argument for the hyperbolic sine,

$$\sinh(\phi) = \pm\sqrt{\frac{\cosh(2\phi)-1}{2}}.$$

Since $\phi > 0$, in this particular case,

$$\sinh(\phi) = \sqrt{\frac{\cosh(2\phi)-1}{2}}.$$

Applying this formula, after simplification with annihilation of the hyperbolic cosine and its inverse, we have that

$$\sinh\left(\frac{1}{2}\cosh^{-1}(u/\tau)\right) = \frac{\sqrt{\mu-\tau}}{\sqrt{2\tau}}.$$

Combining all results together, we obtain that

$$j(\mu,\tau)|_{\beta=\frac{1}{2}} = \frac{1}{\sqrt{2\tau}}\left(\frac{1}{\sqrt{\mu-\tau}} - \frac{1}{\sqrt{\mu+\tau}}\right)\cdot \mathcal{U}(\mu-\tau).$$

Let us simplify the functions f_1 f_2, f_3 and f_4 for $\beta = \frac{1}{2}$. Let us start with consideration of the function f_1. For a general case, this function is given in equation (3.20) and its formula involves the functions i and k. Substituting the simplified formulas i, and k for $\beta = \frac{1}{2}$ we have that

$$f_1|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\pi\sqrt{\omega\sigma}}\int_{-\omega}^{\min\{\omega,\lambda-\sigma\}} \frac{\mathcal{U}(\omega-|u|)\mathcal{U}(\lambda-u-\sigma)\mathcal{U}(\lambda-\sigma+\omega)}{\sqrt{(\omega+u)(\lambda-u-\sigma)}}du.$$

Let us simplify the unit step functions. The unit step function $\mathcal{U}(\omega-|u|)$ tells us that the variable u is restricted by ω and $-\omega$. Since $\omega > 0$, $-\omega < u < \omega$. Next, the unit step function $\mathcal{U}(\lambda-u-\sigma)$ tells us that u is restricted by $\lambda-\sigma$ from above. This tells that $-\omega < u < \min\{\omega, \lambda-\sigma\}$ and $\lambda-\sigma$ should be strictly greater than $-\omega$. Otherwise, the function $f_1(\lambda,\sigma,\omega)$ is undefined if $\lambda-\sigma = -\omega$ or trivial if $\lambda-\sigma < -\omega$. However, all these facts are caught by the integration limits for u in the formula and by the unit step function $\mathcal{U}(\lambda-\sigma+\omega)$. Thus, 2 unit steps functions $\mathcal{U}(\omega-|u|)$ and $\mathcal{U}(\lambda-u-\sigma)$ can be eliminated from the notation of the formula.

Finally,

$$f_1|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\pi\sqrt{\omega\sigma}}\int_{-\omega}^{\min\{\omega,\lambda-\sigma\}} \frac{\mathcal{U}(\lambda-\sigma+\omega)}{\sqrt{(\omega+u)(\lambda-u-\sigma)}}du.$$

Processing the formulas for functions f_{2-4} in a similar way, we have that

$$f_2|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\pi\sqrt{\omega\sigma}}\int_{\omega}^{\lambda-\sigma} \frac{1}{\sqrt{(u-\omega)(\lambda-u-\sigma)}} - \frac{1}{\sqrt{(\omega+u)(\lambda-u-\sigma)}}du\cdot\mathcal{U}(\lambda-\sigma-\omega).$$

$$f_3|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\pi\sqrt{\omega\sigma}}\int_{-\sigma}^{\min\{\sigma,\lambda-\omega\}} \frac{\mathcal{U}(\lambda-\omega+\sigma)}{\sqrt{(\sigma+u)(\lambda-u-\omega)}}du,$$

and

$$f_4|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\pi\sqrt{\omega\sigma}}\int_{\sigma}^{\lambda-\omega} \frac{1}{\sqrt{(u-\sigma)(\lambda-u-\omega)}} - \frac{1}{\sqrt{(\sigma+u)(\lambda-u-\omega)}}du\cdot\mathcal{U}(\lambda-\sigma-\omega).$$

Now we are getting closer to computation of Φ by the Laplace transform. Recall that by equation (3.29),

$$\Phi(\lambda,\sigma) = h_1(\lambda,\sigma) - h_2(\lambda,\sigma) + h_3(\lambda,\sigma) - h_4(\lambda,\sigma),$$

where the functions h_{1-4} are given in the formulas (3.25) – (3.28). Reproducing the system (3.24) which shows how we initially defined the functions h_{1-4}, we have

$$\begin{array}{ll} h_1(\lambda,\sigma) = \sigma^{-\beta}\int_0^\sigma \omega^{\beta+1}R(\omega)f_1(\lambda,\sigma,\omega)d\omega, & h_2(\lambda,\sigma) = \sigma^{-\beta}\int_0^\sigma \omega^{\beta+1}R(\omega)f_2(\lambda,\sigma,\omega)d\omega, \\ h_3(\lambda,\sigma) = \sigma^{-\beta}\int_\sigma^\infty \omega^{\beta+1}R(\omega)f_3(\lambda,\sigma,\omega)d\omega, & h_4(\lambda,\sigma) = \sigma^{-\beta}\int_\sigma^\infty \omega^{\beta+1}R(\omega)f_4(\lambda,\sigma,\omega)d\omega, \end{array}$$

where $R(\omega) = 2\frac{m+1}{m}\eta|_{\lambda=0}$ and $\beta = \frac{1}{m}$.

Next, we will to group together h_1 with h_2 and h_3 with h_4. Deducting the function h_2 from the function h_1 and the function h_3 from the function h_4 we have that

$$h_1(\lambda,\sigma) - h_2(\lambda,\sigma) = \sigma^{-\beta}\int_0^\sigma \omega^{\beta+1}R(\omega)(f_1(\lambda,\sigma,\omega) - f_2(\lambda,\sigma,\omega))d\omega \text{ and}$$
$$h_3(\lambda,\sigma) - h_4(\lambda,\sigma) = \sigma^{-\beta}\int_\sigma^\infty \omega^{\beta+1}R(\omega)(f_3(\lambda,\sigma,\omega) - f_4(\lambda,\sigma,\omega))d\omega.$$

Soon, it will be clear why we are grouping the functions this way.

Let us find $f_1 - f_2$ for $\beta = \frac{1}{2}$. Simple arithmetic and manipulation with unit step functions will show that

$$(f_1 - f_2)|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\pi\sqrt{\sigma\omega}}\left[\int_{-\omega}^{\lambda-\sigma}\frac{\mathcal{U}(\lambda-\sigma+\omega)}{\sqrt{(\omega+u)(\lambda-u-\sigma)}}du - \int_{\omega}^{\lambda-\sigma}\frac{\mathcal{U}(\lambda-\sigma-\omega)}{\sqrt{(u-\omega)(\lambda-u-\sigma)}}du\right].$$

Now we will simplify the integrals in this formula. Let us denote the first integral as I_1. So,

$$I_1 = \int_{-\omega}^{\lambda-\sigma}\frac{du}{\sqrt{(\omega+u)(\lambda-u-\sigma)}}.$$

The following substitution of u by t

$$u = \frac{1}{2}\left((\lambda-\sigma+\omega)\sin(t) + \lambda - \sigma - \omega\right)$$

with simplification gives us the result

$$I_1 = \begin{cases} \pi & \text{if } \lambda - \sigma > -\omega, \\ -\pi & \text{if } \lambda - \sigma < -\omega. \end{cases}$$

Similarly, let us denote the second integral as I_2. So

$$I_2 = \int_{\omega}^{\lambda-\sigma}\frac{1}{\sqrt{(u-\omega)(\lambda-u-\sigma)}}du.$$

The similar substitution

$$u = \frac{1}{2}\left[(\lambda-\sigma-\omega)\sin t + (\lambda-\sigma+\omega)\right]$$

with simplification gives us the similar result

$$I_2 = \begin{cases} \pi & \text{if } \lambda - \sigma > \omega, \\ -\pi & \text{if } \lambda - \sigma < \omega. \end{cases}$$

Combining these results yields

$$f_1|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) - f_2|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\sqrt{\sigma\omega}}\left[\mathcal{U}(\lambda-\sigma+\omega) - \mathcal{U}(\lambda-\sigma-\omega)\right].$$

Let us consider the similar difference of $f_3|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega)$ and $f_4|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega)$. Notice, that the function $f_1(\lambda,\sigma,\omega)$ differs from the function $f_3(\lambda,\sigma,\omega)$ only the by interchange of variables σ and ω. The function $f_2(\lambda,\sigma,\omega)$ differs from the function $f_4(\lambda,\sigma,\omega)$ in the same way. Therefore,

$$f_3|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) - f_4|_{\beta=\frac{1}{2}}(\lambda,\sigma,\omega) = \frac{1}{2\sqrt{\sigma\omega}}\left[\mathcal{U}(\lambda+\sigma-\omega) - \mathcal{U}(\lambda-\sigma-\omega)\right].$$

Then for $\beta = \frac{1}{2}$, $\lambda \geq 0$ and $\sigma > 0$

$$
\begin{aligned}
h_1(\lambda,\sigma) - h_2(\lambda,\sigma) &= \frac{1}{2\sigma}\int_0^{\sigma} \omega R(\omega)\left[\mathcal{U}(\lambda-\sigma+\omega) - \mathcal{U}(\lambda-\sigma-\omega)\right] d\omega \\
&= \frac{1}{2\sigma}\cdot\begin{cases} \int_{\sigma-\lambda}^{\sigma} \omega R(\omega)d\omega & \text{if } \lambda < \sigma, \\ \int_{\lambda-\sigma}^{\sigma} \omega R(\omega)d\omega & \text{if } \sigma \leq \lambda < 2\sigma, \\ 0 & \text{if } \lambda \geq 2\sigma. \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
h_3(\lambda,\sigma) - h_4(\lambda,\sigma) &= \frac{1}{2\sigma}\int_{\sigma}^{\infty} \omega R(\omega)\left[\mathcal{U}(\lambda+\sigma-\omega) - \mathcal{U}(\lambda-\sigma-\omega)\right] d\omega \\
&= \frac{1}{2\sigma}\cdot\begin{cases} \int_{\sigma}^{\lambda+\sigma} \omega R(\omega)d\omega & \text{if } \lambda < 2\sigma, \\ \int_{\lambda-\sigma}^{\lambda+\sigma} \omega R(\omega)d\omega & \text{if } \lambda \geq 2\sigma. \end{cases}
\end{aligned}
$$

Adding the results of these formulas together we have that

$$
\Phi(\lambda,\sigma) = \frac{1}{2\sigma}\begin{cases} \int_{\lambda-\sigma}^{\lambda+\sigma} \omega R(\omega)d\omega & \text{if } \lambda \geq \sigma, \\ \int_{\sigma-\lambda}^{\lambda+\sigma} \omega R(\omega)d\omega & \text{if } \lambda < \sigma. \end{cases}
$$

One more simplification finally gives us that for all $\lambda \geq 0$ and $\sigma > 0$,

$$
\Phi(\lambda,\sigma) = \frac{1}{2\sigma}\int_{|\lambda-\sigma|}^{\lambda+\sigma} \omega R(\omega)d\omega. \tag{4.14}
$$

4.2.2 Method 2. Solving by D'Alembert method after Didenkulova and Pelinovsky

Now, let us solve equation (4.13) by the D'Alembert method, reproducing the approach of Didenkulova and Pelinovsky. Let us introduce the function $\Psi(\lambda,\sigma)$ as

$$
\Psi(\lambda,\sigma) = \begin{cases} \sigma\Phi(\lambda,\sigma) & \text{if } \lambda,\sigma > 0, \\ \lim_{\sigma\to 0+0} \Psi(\lambda,\sigma) & \text{if } \sigma = 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Then, $\Psi_{\lambda\lambda} = \sigma\Phi_{\lambda\lambda}$ and $\Psi_{\sigma\sigma} = \sigma\Phi_{\sigma\sigma} + 2\Phi_{\sigma}$. After multiplying equation (4.13) by σ and substituting the obtained partial derivatives of Ψ into equation we get

$$
\Psi_{\lambda\lambda} - \Psi_{\sigma\sigma} = 0. \tag{4.15}
$$

Using the initial conditions of equation (4.13) given in (2.26) we obtain the following initial conditions for system (4.15)

$$
\Psi|_{\lambda=0} = 0 \text{ and } \Psi_{\lambda}|_{\lambda=0} = \sigma R(\sigma),
$$

where, by formula (3.6), $R(\sigma) = 4g\eta|_{\lambda=0}$.

Using the D'Alembert formula (*Farlow*, 1993) for a semi-finite string (here λ corresponds to time and σ corresponds to the space) with applied initial conditions, we have that for any $\lambda, \sigma > 0$

$$\Psi(\lambda,\sigma) = \begin{cases} \frac{1}{2}\int_{\lambda-\sigma}^{\lambda+\sigma} \omega R(\omega)\omega & \text{if } \lambda \geq \sigma, \\ \frac{1}{2}\int_{\sigma-\lambda}^{\lambda+\sigma} \omega R(\omega)\omega & \text{if } \lambda < \sigma. \end{cases}$$

Simplifying this formula, we have that

$$\Psi(\lambda,\sigma) = \frac{1}{2}\int_{|\lambda-\sigma|}^{\lambda+\sigma} \omega R(\omega)\omega.$$

In fact, this formula is valid even for $\lambda = 0$.

Expressing this solution back in terms of Φ we obtain that that for $\lambda \geq 0$ and $\sigma > 0$

$$\Phi(\lambda,\sigma) = \frac{1}{2\sigma}\int_{|\lambda-\sigma|}^{\lambda+\sigma} \omega R(\omega) d\omega.$$

Note that, this is the same result as the result obtained by the Laplace transformation, represented in the formula (4.14).

Also, notice that the solution of equation (4.13) derived by Didenkulova and Pelinovsky through the D'Alembert formula and given in (*Didenkulova and Pelinovsky*, 2011) is exactly the same as equation (4.14).

Hence, $\Phi(\lambda, \sigma)$, the solution of the linearized equation (2.19) derived through the Laplace transformation, is likely correct for parabolic bays also.

Chapter 5

Relation of the shallow water problem to the wave equation in $\mathbb{R}^n$ space.

In this section we want to discuss an interesting relation: the relation between the waves in an inclined bay by desribed the shallow water equation and the linear radially symmetric wave propagating in the higher dimensional spaces.

5.1 Solution of the wave equation and its spherical mean

Let us consider the initial-value problem for the wave equation in n-dimensional space

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \quad x \in \mathbb{R}^n, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases}$$

where $\Delta u \equiv \sum_{i=1}^n u_{x_i x_i}$.

Now we need to introduce new notation.

Let for a point $x \in \mathbb{R}^n$,

- $B(x,r)$ be the ball of radius r centered at x;
- $\partial B(x,r)$, be the boundary of the ball of radius r centered at x,
- $S.A.(B(x,r))$ be the surface area of $B(x,r)$. Saying, more strictly, this is the measure of $\partial B(x,r)$.

Next, given function $f(x,t)$, let $\overline{f}(x,r,t)$ stand for the average of $f(x,t)$ over the surface $\partial B(x,r)$, defined

$$\overline{f}(x,r,t) = \frac{1}{S.A.(B(x,r))} \int_{\partial B(x,r)} f(y,t)dy.$$

We will call $\overline{f}(x,r,t)$ a **spherical mean** of the function $f(x,t)$. Let for some fixed point $x_0 \in \mathbb{R}^n$ introduce the function $v : \mathbb{R}^2 \to \mathbb{R}$ defined $v(r,t) = \overline{u}(x_0, r, t)$. So, $v(r,t)$ is the average of the wave function of n-dimensioned space $u(x,t)$ over a ball centered at a fixed point x_0 having the radius r.

Then by (*Courant and Hilbert*, 1989) the function $v(r,t)$ solves the initial value problem

$$\begin{cases} v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0, \\ v(r,0) = \overline{u}(x_0, r, 0) = \overline{\phi}(x_0; r), \\ v_t(r,0) = \overline{u}_t(x_0, r, 0) = \overline{\psi}(x_0; r), \end{cases}$$

As one can notice, this problem resembles the problem of the linearized shallow water equation (2.19) with initial conditions (2.26) and with omitted boundary conditions. Let us recall this problem. Written without boundary conditions the linearized shallow water problem has the form:

$$\begin{cases} \Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{m+2}{m\sigma}\Phi_\sigma = 0, \\ \Phi|_{\lambda=0} = 0, \\ \Phi_\lambda|_{\lambda=0} = 2\frac{m+1}{m} g\eta(\sigma, 0). \end{cases}$$

If to replace λ with time and σ with the distance, then the solution of this problem $\Phi(\lambda, \sigma)$ can be viewed as a spherical mean of some wave function $\Psi(x, \lambda)$ at some point $x_0 \in \mathbb{R}^{2\frac{m+1}{m}}$.

Does it look interesting?

Let us recall the particular shallow water problems such as plane beach problem, problem for bays with parabolic and triangular cross-sections which are parameterized by functions $z = |cy|^\infty$, $z = c|y|^2$ and $z = c|y|$ respectively, for $c > 0$. As one can observe from the linearized shallow water equations for each of these problems, their solutions can be viewed as spherical means for the waves propagating in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$ spaces respectively.

If that statement for was extended on spaces with non-integer dimensions, we would probably have that for the most of the other powers, the shallow water problems in these bays could be viewed as spherical means of waves propagating in space with non-integer dimensions.

For example, if we consider the shallow water wave problem for the bay with the cross-section parameterized by cubic function $z = c|y|^3$, then the solution of the linearized shallow water equation for this problem could be viewed as the spherical mean for the wave running in $\mathbb{R}^{2+\frac{2}{3}}$ space.

However, there is a question: how can we interpret the space having the non-integer number of dimensions?

We will restrict our attention to the spherical means of integer dimensional spaces. More precisely - on odd-dimensional spaces.

5.2 Shallow water problems related to the waves in odd-dimensional spaces

Let us consider the class of the linearized shallow water equations

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{2k}{\sigma}\Phi_\sigma = 0, \tag{5.1}$$

where $k \in \mathbb{N}$. Let us recall the linearized shallow water equation (2.19) for a bay with the cross-section, parameterized by the function $z = c|y|^m$, $m \in (0, \infty]$. Then this equation is

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{m+2}{m\sigma}\Phi_\sigma = 0.$$

Solving equation $\frac{m+2}{m} = 2k$ with respect to m we have, that $m = \frac{2}{2k-1}$, for $k \in \mathbb{N}$. So, equation (5.1) describes the shallow water problems in bays, with cross-sections parameterized by power functions $z = c|y|^{\frac{2}{2k-1}}$, for $k \in \mathbb{N}$. At the same time, if $\Phi(\lambda, \sigma)$ is the solution of equation (5.1), then it can be viewed as the spherical mean of wave equation propagating in the $2k+1$-dimensional space. For all natural k these dimensions are odd.

Now it will be clear, why we give attention the shallow water wave problems related to the odd-dimensional space.

Let us consider the shallow water problem for an inclined bay with the cross-section, parameterized by the function $z = c|y|^m$, where $m = \frac{2}{2k-1}$, $k \in \mathbb{N}$. Then the linearized shallow water equation of this problem is equation (5.1). Suppose,

$$\Phi(\lambda, \sigma)|_{\lambda=0} = \phi(\sigma) \text{ and } \Phi_\lambda(\lambda, \sigma)|_{\lambda=0} = \psi(\sigma)$$

are the initial conditions of this equation.

Now, let

$$F = \left(\frac{1}{\sigma}\frac{\partial}{\partial\sigma}\right)^{(k-1)} (\sigma^{2k-1}\Phi(\lambda, \sigma)). \tag{5.2}$$

Then by (*Courant and Hilbert*, 1989) F solves the initial value problem

$$\begin{cases} F_{\lambda\lambda} - F_{\sigma\sigma} = 0, \\ F|_{\lambda=0}(\sigma) = (\frac{1}{\sigma}\frac{\partial}{\partial\sigma})^{k-1}(\sigma^{2k-1}\phi(\sigma)), \\ F_\lambda|_{\lambda=0}(\sigma) = (\frac{1}{\sigma}\frac{\partial}{\partial\sigma})^{k-1}(\sigma^{2k-1}\psi(\sigma)), \\ \lambda \geq 0, \ \sigma > 0, \end{cases} \tag{5.3}$$

and this initial wave problem has the solution defined by the D'Alembert formula for the string on semiaxis (*Farlow*, 1993) (here λ stands for time and σ stands for the length).

Hence, the shallow water problems related to the waves in odd-dimensional space dimensions have very nice property: their linearized shallow water equations defined by (5.1), by the formula (5.2) can be transformed to equations which have explicit solutions given by the D'Alembert formula for semiaxis.

After finding F through the D'Alembert formula and solving the differential equation (5.2) with the boundary conditions given in (2.27), we obtain the explicit formula for $\Phi(\lambda, \sigma)$, the solution of the linearized shallow water problem.

This gives us a different approach of solving the shallow water linear problems at least for such bays, which have cross-section parameterized by $z = c|y|^{\frac{2}{2k-1}}$ for $k \in \mathbb{N}$.

Remark: the shallow water problem for the parabolic bay, considered in the previous section and initially described by equation (4.13) and solved by Didenkulova and Pelinovsky (*Didenkulova and Pelinovsky*, 2011) is the particular case of these problems, for $k = 1$. More over this is the simplest case of the shallow water problems, related to the waves in odd-dimensional spaces. For $k = 1$ the differential equation (5.2) turns into the algebraic equation.

In the next section we consider the particular problem for $k = 2$.

Chapter 6

Sloping bay with the cross-section parameterized by $z = c|y|^{\frac{2}{3}}$, $c > 0$

In this chapter we will consider the shallow water problem in an inclined bay with the cross-section parameterized by the function $z = c|y|^{\frac{2}{3}}$ $c > 0$. This bay has a narrowing bottom and is related to the so-called V-shaped bays. The approximate sketch of its cross-section is in Figure 1.4 d). We also represent it separately in Figure 6.1

By the formula (2.19) the linearized shallow water equation for this problem is

$$\Phi_{\lambda\lambda} - \Phi_{\sigma\sigma} - \frac{4}{\sigma}\Phi_{\sigma} = 0 \tag{6.1}$$

with the initial and boundary conditions given respectively by (2.26) and (2.27)

$$\Phi(\lambda,\sigma)|_{\lambda=0} = 0, \quad \Phi_{\lambda}(\lambda,\sigma)|_{\lambda=0} = R(\sigma) = 5g\eta(\lambda,\sigma)|_{\lambda=0}, \tag{6.2}$$

$$|\Phi(\lambda,\sigma)\,|_{\sigma=0}\,| < \infty, \quad \Phi_{\sigma}(\lambda,\sigma)|_{\sigma=0} = 0, \quad \lim_{\sigma\to\infty}\Phi(\lambda,\sigma) = 0. \tag{6.3}$$

6.1 Derivation of $\Phi(\lambda,\sigma)$, the solution of the linearized shallow water equation (6.1) with given initial and boundary conditions

Notice that this shallow water problem is related to the problem of the wave propagating in the odd-dimensional space, described in the previous chapter. The partial differential equation (6.1) is a particular case of equation (5.1) for $k = 2$ and as we showed referring on (*Courant and Hilbert*, 1989), its solution $\Phi(\lambda,\sigma)$ can be viewed as the spherical mean of the wave propagating in $\mathbb{R}^5$ space.

Also, recall, that at the end of chapter 5, using properties of spherical means from the source (*Courant and Hilbert*, 1989) we developed the method of solving equation (5.1) for $k \in \mathbb{N}$ with known initial and boundary conditions. Let us apply this method for solving equation (6.1) with the initial conditions (6.2) and boundary conditions (6.3).

For $k = 2$ the differential equation (5.2) become the nonhomogeneous ordinary differential equation of the first order

$$\Phi_{\sigma} + \frac{3}{\sigma}\Phi_{\sigma} = \frac{F}{\sigma^2}, \tag{6.4}$$

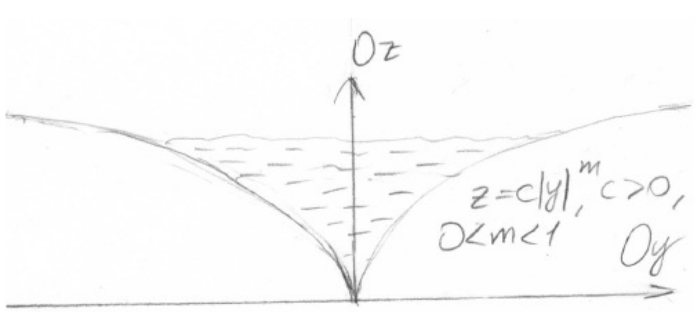


Figure 6.1: Approximate sketch of $z = y^{2/3}$

where $F(\lambda,\sigma)$ by systems (6.2) and (5.3) is the solution of the initial value problem

$$\begin{cases} F_{\lambda\lambda} - F_{\sigma\sigma} = 0, \\ F(\lambda,\sigma)|_{\lambda=0} = 0, \\ F(\lambda,\sigma)_{\lambda}|_{\lambda=0} = 3\omega R(\omega) + \omega^2 R'(\omega), \\ \lambda \geq 0,\ \ \sigma > 0. \end{cases} \tag{6.5}$$

Then by the D'Alembert formula for a semifinite string,

$$F(\lambda,\sigma) = \frac{1}{2}\int_{|\lambda-\sigma|}^{\lambda+\sigma} 3\omega R(\omega) + \omega^2 R'(\omega) d\omega. \tag{6.6}$$

Now, let us come back to the differential equation (6.4). As one can see, the solution of this equation, $\Phi(\lambda,\sigma)$ has the form

$$\Phi(\lambda,\sigma) = \frac{c(\lambda,\sigma)}{\sigma^3},$$

where $c(\lambda,\sigma)$ is a function of two arguments λ and σ satisfying

$$\frac{1}{\sigma^3}\frac{\partial}{\partial\sigma}c(\lambda,\sigma) = \frac{F(\lambda,\sigma)}{\sigma^2}.$$

This implies that

$$c(\lambda,\sigma) = c_2(\lambda) + \frac{1}{2}\int_0^{\sigma} \sigma' F(\lambda,\sigma')d\sigma',$$

where $c_2(\lambda)$ is some function of one variable λ.

Next, multiplying both parts of this formula by σ^3, and letting $\sigma \to 0$ we have that

$$c_2(\lambda) = \lim_{\sigma\to 0}\sigma^3\left[\Phi(\lambda,\sigma) + \int_0^{\sigma} \sigma' F(\lambda,\sigma')d\sigma\right].$$

Next, notice that using integration by parts, the formula (6.6) can be simplified to

$$F(\lambda,\sigma) = \frac{1}{2}\left[\left[\omega^2 R(\omega)\right]|_{|\lambda-\sigma|}^{\lambda+\sigma} + \int_{|\lambda-\sigma|}^{\lambda+\sigma} \omega R(\omega)d\omega\right]. \tag{6.7}$$

Since $\Phi(\lambda,\sigma)$ and sigma $R(\sigma)$ are bounded $\sigma = 0$ (6.3) it gives us that $c_2(\lambda) \equiv 0$.

Therefore, for $\lambda \geq 0$ and $\sigma > 0$,

$$\Phi(\lambda,\sigma) = \frac{1}{\sigma^3}\int_0^{\sigma} \sigma' F(\lambda,\sigma')d\sigma', \tag{6.8}$$

where F is given in the formulas (6.6) and (6.7). In order to emphasize that we consider Φ only for nonnegative values of σ and λ we can extend this function for negative λ and σ as 0.

Let us consider Φ for $\sigma = 0$. Notice that the formula (6.8) becomes undetermined for this σ. Assuming that $\Phi(\lambda,\sigma)$ is continuous for $\sigma > 0$, we can define it as

$$\Phi(\lambda,\sigma)|_{\sigma=0} = \lim_{\sigma\to 0}\frac{\int_0^{\sigma}\sigma' F(\lambda,\sigma')d\sigma'}{\sigma^3}.$$

Applying the L'Hospital rule twice, we have that for any $\lambda \geq 0$

$$\Phi(\lambda,\sigma)|_{\sigma=0} = \lambda R(\lambda) + \frac{1}{3}\lambda^2 R'(\lambda). \tag{6.9}$$

Finally we have that the solution of equation (6.1), with initial conditions (6.2) and boundary conditions (6.3) is

$$\Phi(\lambda,\sigma) = \begin{cases} \sigma^{-3}\int_0^\sigma \sigma' F(\lambda,\sigma')d\sigma' & \lambda \geq 0,\ \sigma > 0, \\ \lambda R(\lambda) + \frac{1}{3}\lambda^2 R'(\lambda) & \lambda \geq 0,\ \sigma = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{6.10}$$

where by (6.7)

$$F(\lambda,\sigma) = \frac{1}{2}\left[\left[\omega^2 R(\omega)\right]\big|_{|\lambda-\sigma|}^{\lambda+\sigma} + \int_{|\lambda-\sigma|}^{\lambda+\sigma} \omega R(\omega)d\omega\right]$$

and

$$R(\sigma) = 5g\eta(\lambda,\sigma)|_{\lambda=0}.$$

6.2 Comparison of the obtained solution Φ for the bay described by $z = c|y|^{\frac{2}{3}}$ with the solution obtained through the Laplace transform

We want to be sure that the solution of the potential $\Phi(\lambda,\sigma)$ for this bay obtained through the D'Alembert formula and given in equation (6.10) is correct. We will compare it with the solution obtained through the Laplace transform, given by equations (3.29) and (3.34). We will do it numerically. We will compute it for a particular example by both ways and compare the results.

Let us consider the wave on a sloping bay with the slope $\alpha = 0.01$ and a cross-section parameterized by $z = c|y|^{\frac{2}{3}}$, $c > 0$ (As one could notice, the absolute value of c is not important). Let the initial profile of the wave be given by the formula

$$\eta(H) = \begin{cases} 0.1 \cdot \cos^5\left(\frac{\pi}{2} \cdot \frac{H-100.1}{99}\right)[m] & \text{if } 1.1[m] < H < 199.1[m], \\ 0 & \text{otherwise}, \end{cases}$$

where $H(x,t)$ is the maximal water depth in a cross-section, which is taken for a fixed physical point x and is orthogonal to the wave front. We chose a high power of the cosine because we want that the function Φ would have continuous derivatives up to the third order and because a wave with this initial profile has nice runup properties. A picture of this wave is showed in Figure 6.2 and in Figure 6.3, and the picture of the potential Φ computed through the D'Alembert solution is showed in Figure 6.4.

As one can see, the initial wave is the wave with the amplitude of $0.1m$ (10 cantimeters), and a length about 16 kilometers.

Let us denote

- $\Phi^L(\lambda,\sigma)$-be the solution computed by the formula (3.34), obtained through Laplace transform

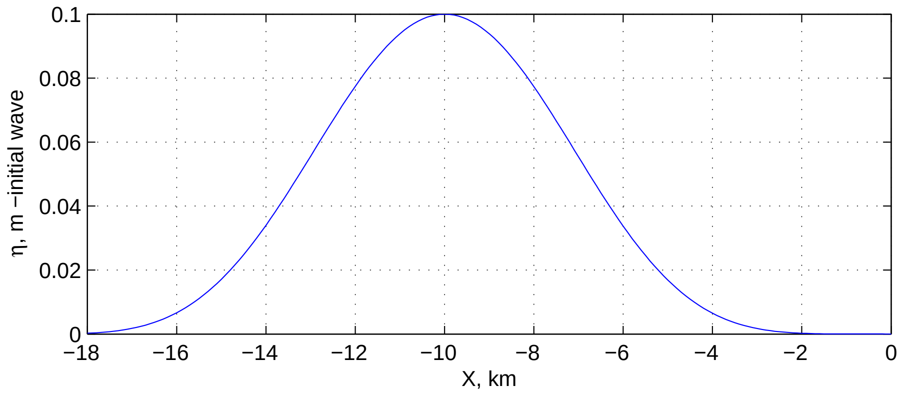


Figure 6.2: Initial vertical water displacement, $\eta(x)$

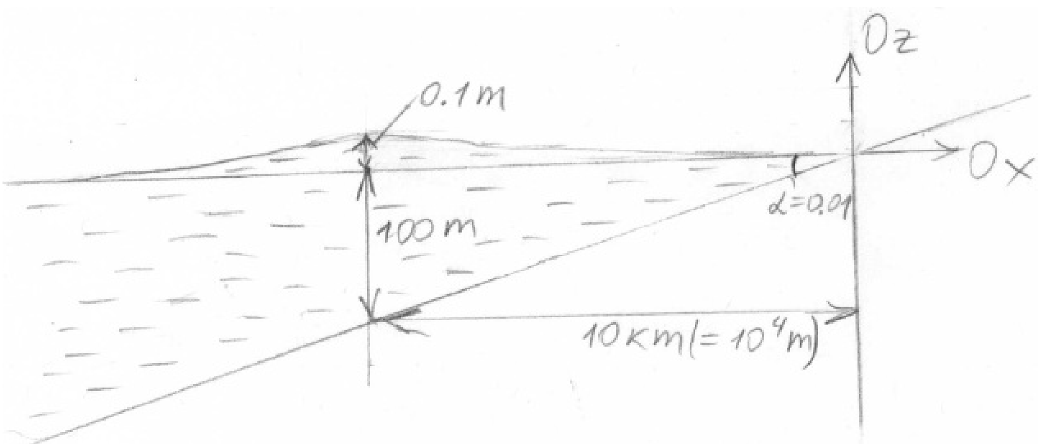


Figure 6.3: Initial wave and a bay in the offshore direction

- $\Phi^D(\lambda, \sigma)$ be the solution computed by the formula (6.10) , obtained through the D'Alembert formula and developed in this chapter

As one can see, Φ has the large values on that part of $\lambda \times \sigma$ grid, where σ is small. We will compute the set of values $\Phi^L(\lambda_m, \sigma_n)$ and $\Phi^D(\lambda_m, \sigma_n)$ for $\lambda_m \in \{1, 2, 3, ..., M\}$ and $\sigma_n \in \{1, 2, ..., N\}$ where $M = 200$, $N = 10$ and then find the average of $\Phi^D$and of difference $\Phi^L - \Phi^D$ on the grid. We will to do it through computation of these expressions

$$S = \frac{1}{M \cdot N} \sum_{m=1}^{M} \sum_{n=1}^{N} \left|\left|\Phi^D(\lambda_m, \sigma_n)\right|\right|_2 \text{ and } \Delta S = \frac{1}{M \cdot N} \sum_{m=1}^{M} \sum_{n=1}^{N} \left|\left| \left[\Phi^L(\lambda_m, \sigma_n) - \Phi^D(\lambda_m, \sigma_n)\right] \right|\right|_2$$

and estimation of their ratio.

As the numerical results showed, for $M = 200$ and $N = 10$,

$$S \approx 7 \text{ and } \Delta S = 1.2 \cdot 10^{-4},$$

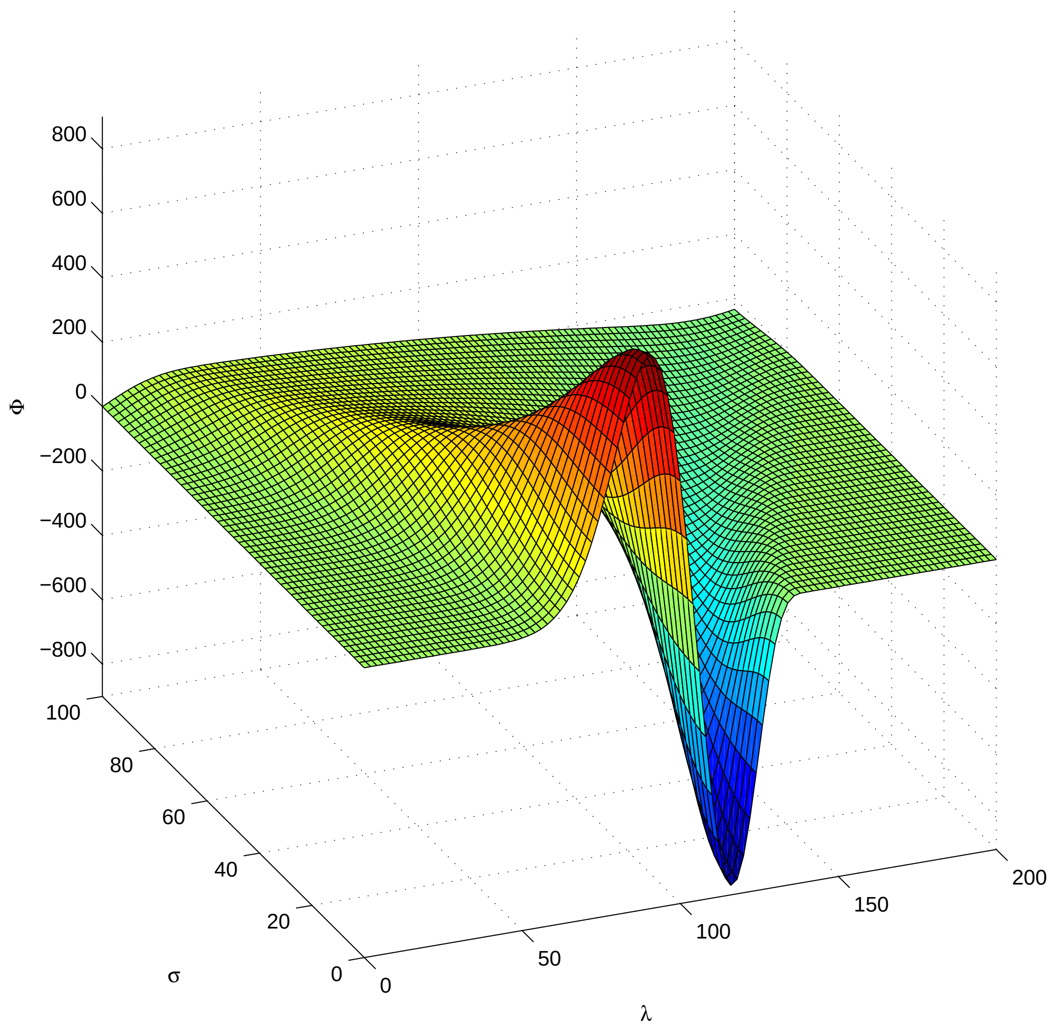


Figure 6.4: Potential Φ for the wave displayed in Figure 6.2 and in Figure 6.3

and then their ratio is

$$\frac{\Delta S}{S} \approx 1.7 \cdot 10^{-5}.$$

So the difference of the solutions expressed through very different formulas and computed on a grid of 2000 values is relatively very small and then with the large likelihood we can say that this difference was produced by numerical errors.

Therefore, both solutions, obtained through the D'Alembert formula and Laplace transformation are most likely both correct.

Remark: recall that the solution of the linearized shallow water equation obtained by the Laplace transform has 2 representations which differ by the order of integration. However, we used only one of these formulas, paying attention to the order of integration. We did not consider another order of integration represented in the formula (3.29), because this formula appeared to be much less numerically stable.

6.3 Physical characteristics of a wave in the bay and partial derivatives of $\Phi(\lambda,\sigma)$

The transformation from coordinates back to physical characteristics is given in the system (2.24). Since this bottom profile is parameterized by power function $z = |y|^{2/3}$, substituting $m = 2/3$ into this system gives us the following formulas:

$$\begin{array}{lll} u = \frac{\Phi_\sigma}{\sigma}, & t = \frac{\lambda - u}{\alpha g}, & H = \frac{\sigma^2}{10g}, \\ \eta = \frac{1}{2g}\left[\frac{2}{5}\Phi_\lambda - u^2\right], & x = \frac{1}{2\alpha g}\left(\frac{2}{5}\left[\Phi_\lambda - \frac{\sigma^2}{2}\right] - u^2\right), & \lambda,\, \sigma \geq 0, \end{array} \tag{6.11}$$

where the partial derivatives can be computed from the formula (6.10) either numerically or explicitly. Note, that unlike the formula for the solution obtained through Laplace transform, this formula is easier to work with. Here are some explicit formulas and limits derived from equation (6.10):

$$\Phi_\sigma(\lambda,\sigma) = \begin{cases} \sigma^{-2}F(\lambda,\sigma) - 3\sigma^{-1}\Phi(\lambda,\sigma) & \lambda \geq 0, \sigma > 0, \\ 0 & \sigma = 0, \end{cases} \tag{6.12}$$

where $F(\lambda,\sigma)$ is given in equations (6.6) and (6.7),

$$\Phi_\lambda(\lambda,\sigma) = \begin{cases} \sigma^{-3}\int_0^\sigma \sigma' F_\lambda(\lambda,\sigma)d\sigma' & \lambda \geq 0,\ \sigma > 0, \\ R(\lambda) + \frac{5}{3}\lambda R'(\lambda) + \frac{1}{3}\lambda^2 R''(\lambda) & \lambda \geq 0, \sigma = 0, \end{cases} \tag{6.13}$$

where

$$F_\lambda(\lambda,\sigma) = \Theta(\lambda+\sigma) - \Theta(|\lambda-\sigma|)\mathrm{sign}(\lambda-\sigma)$$

with

$$\Theta(\xi) = \frac{1}{2}\left[3\xi R(\xi) + \xi^2 R'(\xi)\right].$$

So, we have the explicit formulas for partial derivatives. They give us explicit formulas for all physical coordinates for $\lambda \geq 0$, and $\sigma > 0$. The next asymptotic formula

$$u(\lambda,0) = \frac{1}{15}\left(8R'(\lambda) + 7\lambda R''(\lambda) + \lambda^2 R'''(\lambda)\right)$$

allows us to explicitly obtain the physical results for the shoreline.

Remark: the explicit formulas for $\sigma = 0$ are especially valuable since all the explicit formulas for Φ and its derivatives defined for positive σ are undetermined at $\sigma = 0$. Consequently, if we try to obtain through them the physical characteristics for a shoreline, taking σ to be very small, these results can be numerically inaccurate.

Chapter 7
Practical experiments

We will consider a fixed initial vertical water displacement and compare its runup characteristics for the following different bays:

(1) plane beach bay,

(2) parabolic bay, an inclined bay with the cross-section parameterized by $z = c|y|^2$, $c > 0$,

(3) triangular bay, an inclined bay with the cross-section parameterized by $z = c|y|$, $c > 0$,

(4) an inclined bay with the cross-section parameterized by $z = c|y|^{\frac{2}{3}}$,

(5) an inclined bay with the cross-section parameterized by $z = c|y|^{\frac{1}{2}}$.

Sketches of the bays are shown in Figure 1.4.

We will investigate the wave with the same initial conditions as we did it in the previous chapter: Let the slope of the bottom in the offshore direction be equal $\alpha = 0.01$ and let the initial water displacement be defined by the function

$$\eta(H) = \begin{cases} 0.1 \cdot \cos^5\left(\frac{\pi}{2} \cdot \frac{H-100.1}{99}\right)[m] & \text{if } 1.1[m] < H < 199.1[m], \\ 0 & \text{otherwise,} \end{cases}$$

where $H(x,t)$ is the maximal water depth in the cross-section, taken at a fixed physical point x and is orthogonal to the wave front.

The initial shape of the wave is given in Figure 7.1 and the view of the bottom of a bay in the offshore direction is given in Figure 7.2.

As one can see, this wave is about 16 kilometers ($\approx$ 10miles) long which starts about 2 kilometers offshore and has the maximal height of 0.1 meter, (10 centimeters, or $\approx$ 0.33 feet).

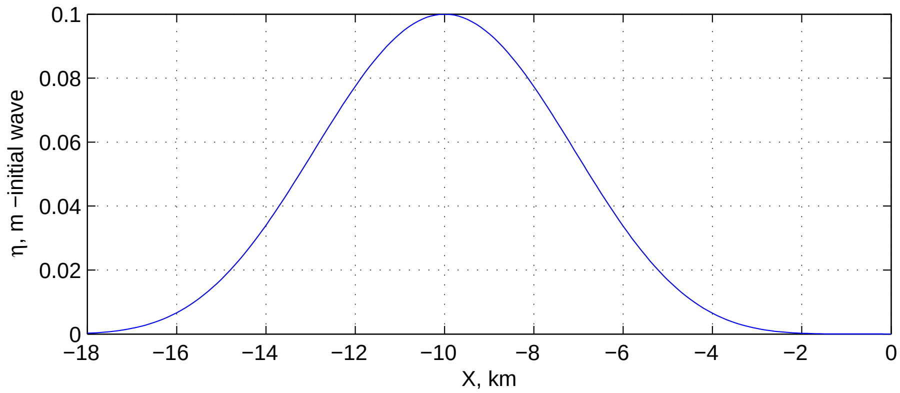


Figure 7.1: Initial vertical water displacement, $\eta(x)$

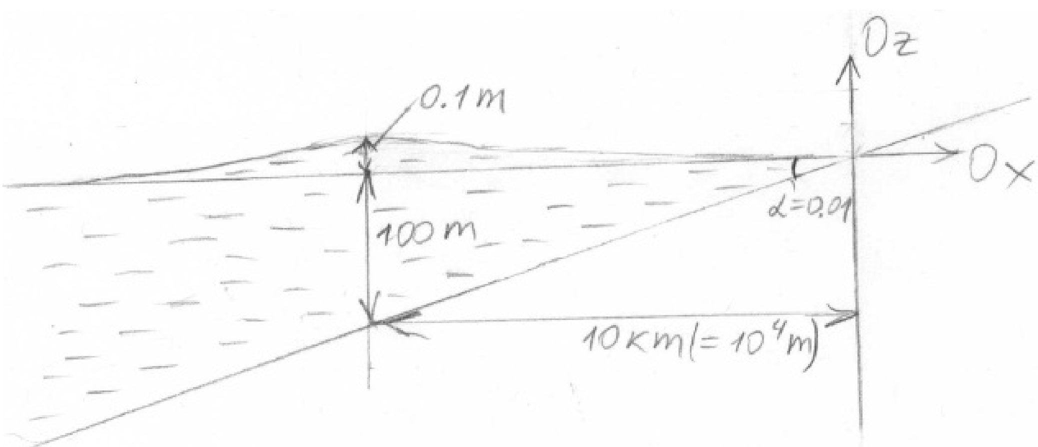
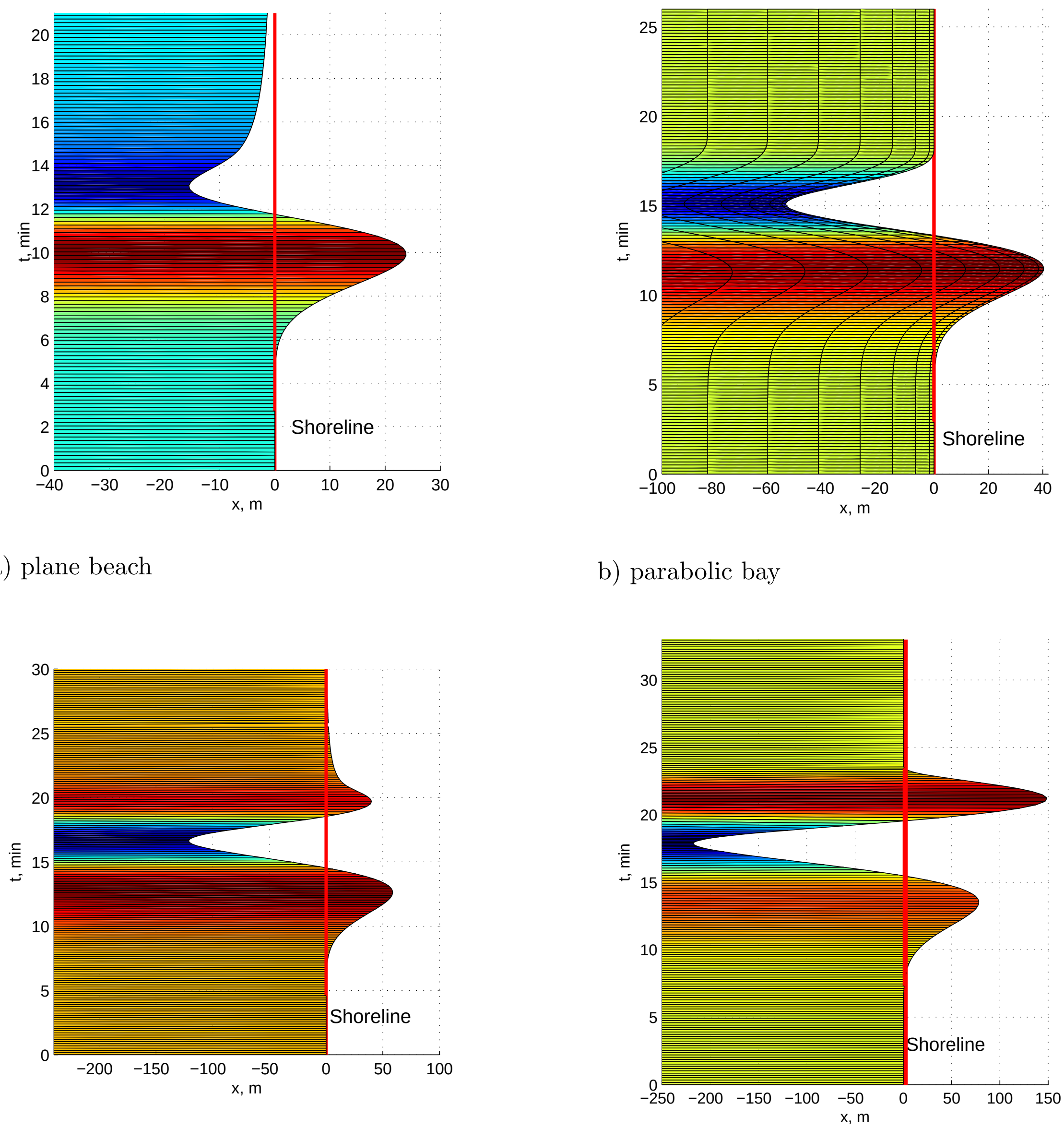


Figure 7.2: Initial wave and a bay in the offshore direction

Here, Figure 7.3 and Figure 7.4 show runups and rundowns of that wave for five different bays. As one can see, there is a noticeable difference: the narrower cross-section bay has, the greater the runup and rundown. If for a plane beach the wave runs inland up to 24 meters, then for the parabolic bay (bay with the cross-section $z = c|y|^2$, $c > 0$) the same wave runs already about up 40 meters; for the triangular bay this distance reaches about 60 meters, but for 2/3 bay and 1/2 bays (bays with the cross-sections $z = c|y|^{\frac{2}{3}}$, $c > 0$ and $z = c|y|^{\frac{1}{2}}$, $c > 0$ respectively) the maximal runup is huge - it is up to 150 meters in the 2/3 bay and over 300 meters for the 1/2 bay!



a) plane beach

b) parabolic bay

c) triangular bay

d) 2/3-bay

Figure 7.3: Horizontal water displacement near the shore

Figure 7.5 and Figure 7.6 show the vertical change of the water level on the shore and Figure 6.3 shows that initial height of the wave is $0.1m$. As one can see from the pictures on Figure 7.5 and from Figure 7.6, if in the plane beach the wave grows up to 2.4 times, then in the parabolic bay and the triangular bay it grows up to 4 and 6 times. In the 2/3-bay and in the 1/2-bay this wave grows up to 15 times and approximately up to 35 times! So, the wave in the 2/3-bay and especially in the

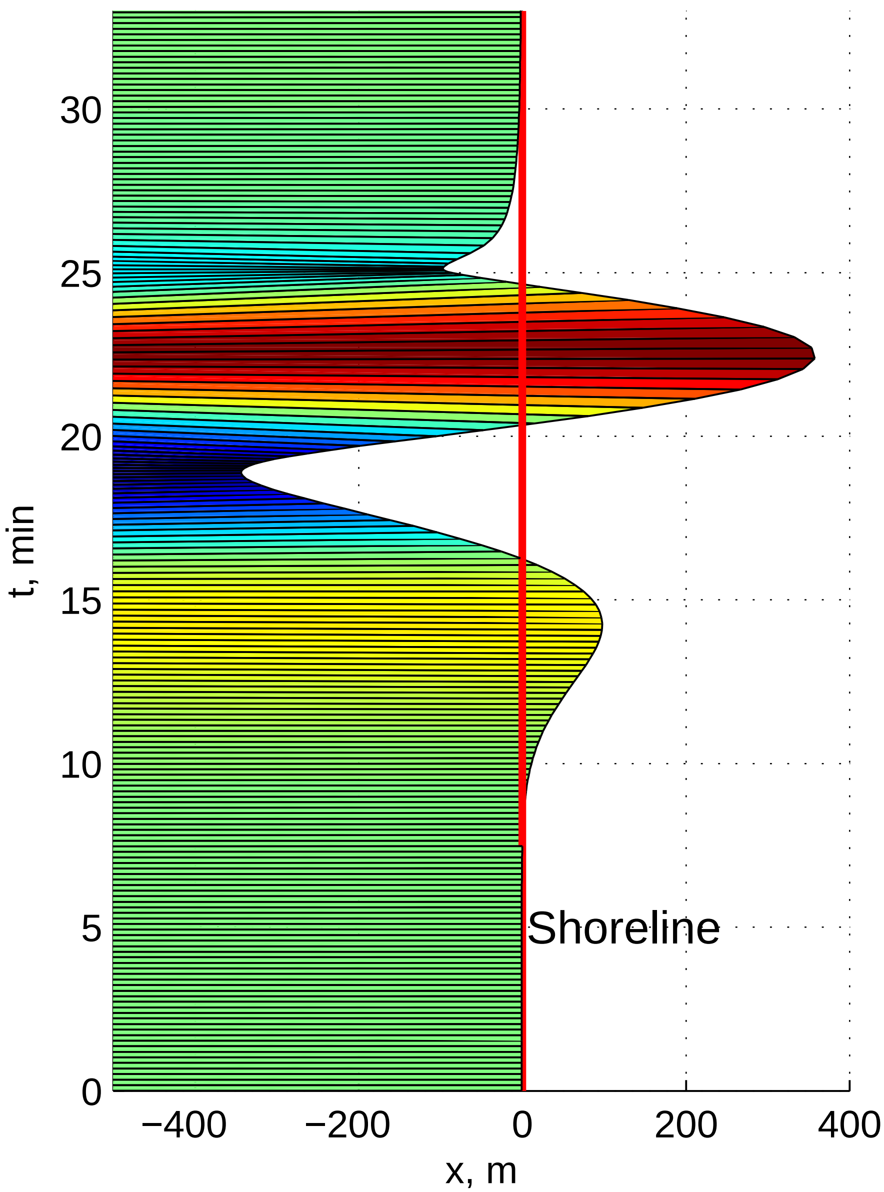
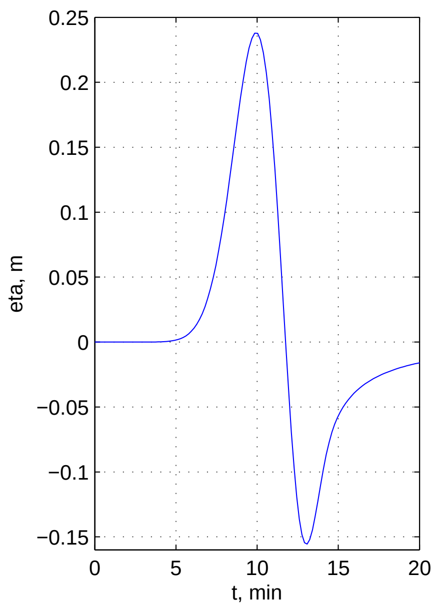


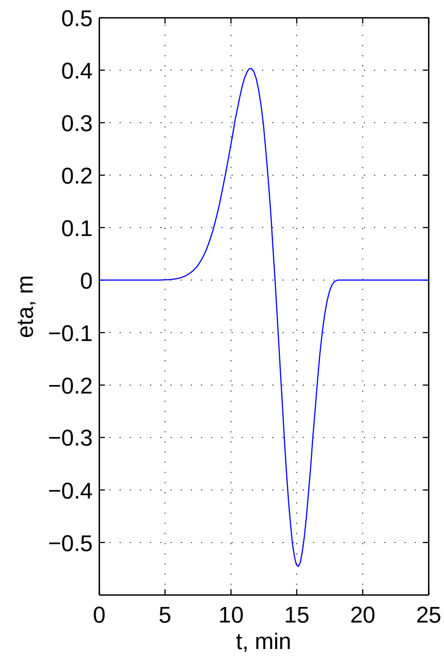
Figure 7.4: Horizontal water displacement near the shore for the 1/2-bay

1/2-bay has a huge height! It says that waves in V shaped bays are especially high and dangerous!

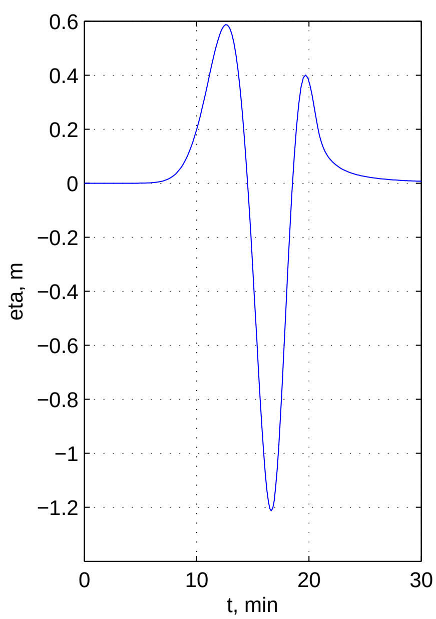
The wave in the triangular, 2/3 and 1/2 bays have more complicated behavior than in the case of the plane beach and parabolic bay. Unlike in the plane beach and parabolic bay the wave runs up twice! It means that in these cases the water after runup and rundown, the water comes back.



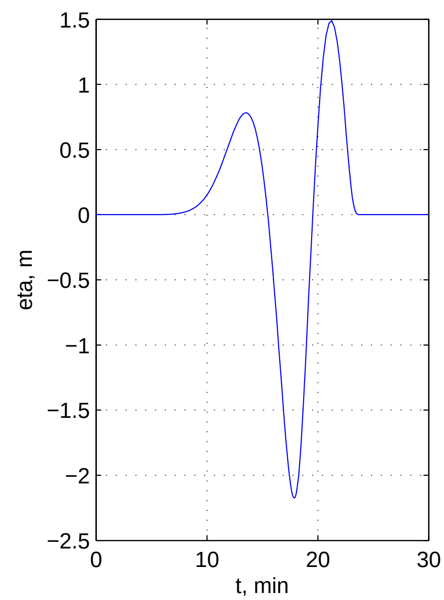
a) plane beach



b) parabolic bay



c) triangular bay



d) 2/3-bay

Figure 7.5: Vertical water displacement near the shore

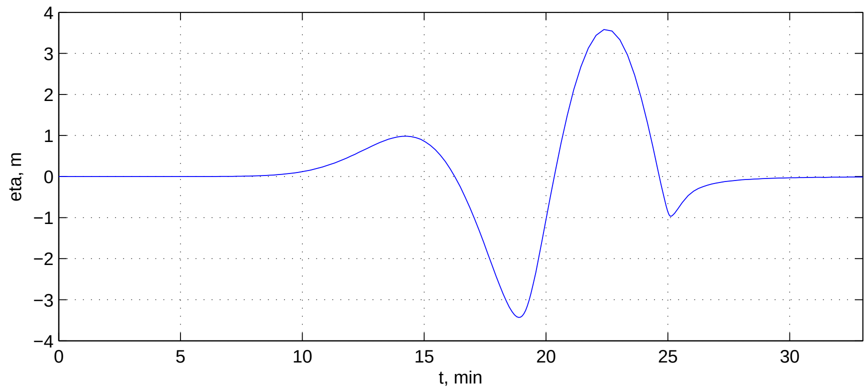


Figure 7.6: Vertical water displacement near the shore for the 1/2-bay

People living around bays which have a form close to those shapes should be aware of this!

Conclusion

In this paper we showed that the physical characteristics of a uniformly approaching tsunami wave, propagating in a uniformly sloping bay with a cross-section parameterized by a power function can be obtained through explicit formulas for any positive power. For comparison, such solutions were previously obtained only for two particular cases of sloping bays: the parabolic bay and the plane beach bay.

The nonlinear system describing the tsunami wave was transformed into a linear equation by hodograph type transformation. Then through the Laplace transform the explicit solution of the linear equation was obtained in terms of elementary functions. This can help researchers to reduce the computational expenses required to investigate tsunami waves propagating in these bays and explicit formulas make the analysis of the solutions easier.

However, the solution of the linear equation has certain disadvantages. The first disadvantage is that the general formula representing the solution has an unsuitable representation in the form of double improper integrals. This complicates the analysis of this analytic solution. The numerical computation of the analytic solution can appear to be unstable and in general it needs to be optimized and can require advanced integration techniques. The second disadvantage of the solution is that the explicit computation of the partial derivatives is also complicated in general, and the computation of physical variables involves partial derivatives of this solution. Consequently, in this paper we did not try to explicitly obtain the partial derivatives of the solution of the linearized problem in the general case. Instead, we have mostly done the differentiation approximately, through the discretized derivatives.

Despite these problems, in bays which have cross-sections parameterized by some particular powers our solution can be greatly simplified and this can make investigation of tsunami in these bays simpler. Therefore, our obtained solution is a very useful tool for further research of the bays that have bottom shapes parameterized by power functions.

The second important fact discussed in the paper is that for each bay corresponding to the power function $z = c|y|^{\frac{2}{2k-1}}$, where $c > 0$ and k is a natural number, the linearized shallow water equation can also be solved through the D'Alembert formula. This let us generalize the approach of Didenkulova and Pelinovsky (*Didenkulova and Pelinovsky*, 2011) who used the D'Alembert formula to solve the linearized shallow water equation for sloping a bay with parabolic cross-section. In fact, their solution has a very simple and neat formula.

We solved through the D'Alembert formula the linearized shallow water equation for a bay corresponding to the power of 2/3, (we call this bay 2/3-bay). The nice property of this solution is that while this formula was more complicated than the formula for a parabolic bay, it had still a pretty suitable representation. Moreover, the asymptotical analysis, explicit differentiation and numerical realization of the formula obtained through the D'Alembert method were simple. In contrast, the same solution for the 2/3-bay obtained through the Laplace transform, looked very unsuitable and required long step-by-step simplification before it could be used to study the wave behavior. Consequently, in the investigation of a wave in a 2/3-bay, we were using the solution of the linearized problem in the form which was obtained by the D'Alembert method.

However, for bays corresponding to powers 2/7, 2/9, etc. the method of seeking the solution of the linearized problem through the D'Alembert method is most likely not applicable. For each bay corresponding to a power $\frac{2}{2k-1}$, where k is a natural number, seeking the solution through the D'Alembert method requires a researcher to solve a non-homogeneous differential equation of $k-1$ order with varying coefficients. However, a nonhomogeneous differential equation with varying coefficients for the orders greater than 2 does not necessarily have an explicit solution. This fact makes the practical applicability of the D'Alembert method for shallow water equations very restricted.

Finally, using these results we investigated the behavior of nonlinear shallow water waves in 1/2-bay, 2/3-bay, a triangular bay, a parabolic and a plane beach bay. In particular, we fixed a wave with the same initial water displacement, made it propagate in all these bays, and compared its behavior in each bay. The investigation of the wave in the 1/2-bay, in the 2/3-bay, and in the triangular bay were of the special interest. As one could see from pictures, the amplitudes of runup and rundown of a wave in the triangular bay, in the 2/3-bay, and in the 1/2-bay, were much higher in comparison with similar amplitudes in the parabolic bay, and in the plane beach bay. The runup and run-down of the wave in the 1/2-bay were especially high. The behaviors of the wave near shores of the 1/2-bay, 2/3-bay and of the triangular bay were more complicated than its behavior near the shores of the parabolic and the plane beach bays. In the plane beach bay and in the parabolic bay the wave first ran up, second, it ran down and third it went offshore. However, both in the triangle bay and in the 2/3-bay after similar runup and rundown the wave had runup second time and only after that it traveled offshore. The behavior of the wave in the 1/2-bay was even more complicated! The wave had first runup, first run-down, second runup and second rundown, and only after that the wave did a travel offshore. It seems that the "narrower" the bottom of the bay becomes (the parameter m for the function $z = c|y|^m$, where $c > 0$ and $m \in (0, \infty]$ parameterizing the cross-section becomes less), the more complicated the behavior of the wave becomes! In particular this causes higher runup and a wave breaks in these bays easier.

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Appendix A

Modified Bessel Functions and their asymptotic formulas

A.1 Modified Bessel functions

The function $I_\nu(x)$ is the Modified Bessel Function of the first kind and it equals

$$I_\nu(x) = i^{-\nu} J_\nu(ix),\ x > 0,$$

where $J_\nu(x)$ is the Bessel function of the first kind. Represented in the form of series this function equals to

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

The function $K_\nu(x)$ is the Modified Bessel Function of the first kind and equals to

$$K_\nu(x) = \begin{cases} \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)},\ x > 0,\ \nu \notin \mathbb{Z}, \\ \frac{\pi}{2} \lim_{\beta\to\nu} \frac{I_{-\beta}(x) - I_\beta(x)}{\sin(\pi\beta)},\ x > 0,\ \nu \in \mathbb{Z}. \end{cases}$$

A.2 The asymptotic behavior of the Modified Bessel functions

If ν is fixed and $x \to 0$:

$$I_\nu(x) \sim \left(\frac{x}{2}\right)^\nu / \Gamma(\nu+1),\ (\nu \neq -1, -2, ...),$$

and

$$K_\nu(x) \sim \begin{cases} -\ln x & \nu = 0, \\ \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}x\right)^{-\nu} & (\nu > 0). \end{cases}$$

If ν is fixed and $x \to +\infty$,

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \text{ and } K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-z}.$$